

N 70 22574

NASA CR 109249

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

*Technical Report 32-1430*

*Calculation of Scattered Patterns  
From Asymmetrical Reflectors*

*Arthur C. Ludwig*

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**JET PROPULSION LABORATORY  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
PASADENA, CALIFORNIA**

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Prepared Under Contract No. NAS 7-100  
National Aeronautics and Space Administration

## **Preface**

The work described in this report was performed by the Telecommunications Division of the Jet Propulsion Laboratory.

## **Acknowledgment**

This report is based upon a dissertation presented to the faculty of the Graduate School, University of Southern California, in partial fulfillment of the requirements for the degree of Doctor of Philosophy (Electrical Engineering).

The author wishes to acknowledge the encouragement and support of Zohrab Kaprielian, guidance committee chairman; Charles Weber and W. V. T. Rusch, dissertation committee co-chairmen; and Alan Schumitzky and Lloyd Welch, dissertation committee members. Also, it should be noted that W. V. T. Rusch did much of the original work on numerical evaluation of scattered patterns using the physical-optics technique upon which the author's research is based.

The author also wishes to acknowledge the assistance of Gerald Levy and Dan Bathker of JPL and, in particular, Phil Potter, whose work on spherical-wave theory provided the background for the author's research in that area.

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## **Abstract**

A computer program for evaluating the electromagnetic field pattern of a known arbitrary incident field scattered from a perfectly conducting reflector of arbitrary shape is presented.

It is shown that the commonly used assumption of far-field behavior for the incident field leads to poor results in some cases; an exact method is developed in which a spherical-wave expansion is used to represent the incident field, and it is shown that this method yields the correct result in these cases.

The scattering surface is represented by a Fourier type of expansion in one variable, where the coefficients are specified as tabular functions of the remaining variable. A tilted figure of revolution is a case of practical importance, and it is shown that, for small tilt angles, three Fourier components are usually sufficient to represent the surface.

A new nonlinear integration technique is used that is four to eight times faster than Simpson's rule integration, under specified conditions. Other techniques to maximize program efficiency are described, and a comparison is given that indicates the program to be 11 to 19 times faster than a similar existing program.

The subreflector and main reflector patterns of an 85-ft-diam Cassegrainian antenna are computed, using both the far-field approximation and a spherical-wave expansion to represent the incident fields. The results are compared, and it is shown that the spherical-wave expansion of the subreflector pattern agrees in remarkable detail with an experimentally measured pattern.

Exemplifying the general capabilities of the program, the patterns of a paraboloid with a displaced feed are computed for three cases, and the results are compared with approximate theory.

# Calculation of Scattered Patterns From Asymmetrical Reflectors

## I. Introduction

The physical-optics technique has been used with great success to compute the scattering from perfectly conducting surfaces, such as the subreflector of a Cassegrainian antenna (Ref. 1). In fact, the technique has yielded such accurate results that computed patterns have begun to replace a good deal of experimental data in antenna development and analysis (Refs. 2 and 3).

A major limitation of the physical-optics technique is that it requires the evaluation of a double integral with a rapidly oscillating integrand that typically results in the need for large blocks of stored data and increased computer time. In some cases, with special symmetry conditions, one integration may be performed analytically (Ref. 4). The program described in this report was developed so that the broad class of scattering problems, which do not necessarily have any symmetries, could be resolved in the most efficient manner possible. The great majority of computer time and storage requirements may be directly attributed to the numerical evaluation of the integral; therefore, the major contribution towards efficiency in this program is the application of a new integration technique (Ref. 5).

The case of scattering from an infinite plane reflector is one of the few electromagnetic scattering problems that has a simple known solution; the resulting field is, of course, identically the reflection of the incident field. An earlier version of the computer program described herein implicitly assumed far-field behavior for the incident-field pattern. When the program was checked, with the plane reflector problem as a test case, it was found that the computed scattered pattern was not the same as the incident pattern and that the result was dependent upon the wavelength of the incident field. (Data for this test case are presented in detail in Section III.)

Of the several possible explanations for this anomaly, it was hypothesized that the far-field behavior of the incident fields was the cause. A technique based upon spherical-wave theory (which yields an exact representation of the incident fields) was developed, and the application of this technique eliminated the error.

The spherical-wave representation of the incident fields is in principle completely general. Also, a general representation of the scattering surface was used in the

development of the program. Practical considerations required that some of this generality be sacrificed in the actual programming, but the theory could easily be applied to extend the results if required.

The only important assumption made in the development of this program was the physical-optics approximation in which the currents on the reflecting surface were assumed to be determined by the incident magnetic field. The validity of this procedure is discussed in Section II; however, the agreement between experimental and computed results presented in Section VI and elsewhere (see Refs. 1-3) is, in fact, the strongest demonstration that the physical-optics approximation is a useful and valid engineering method.

## II. The Physical-Optics Technique

If the currents  $\mathbf{K}$  induced on a surface  $S$  by incident fields  $\mathbf{E}_i$  and  $\mathbf{H}_i$  can be determined, then the scattered fields  $\mathbf{E}_s$  and  $\mathbf{H}_s$  caused by these currents are given by

$$\mathbf{E}_s(P) = -\frac{j}{4\pi\omega\epsilon} \int_S [(\mathbf{K} \cdot \nabla) \nabla + k^2 \mathbf{K}] \frac{e^{-jkr}}{r} dS \quad (1a)$$

and

$$\mathbf{H}_s(P) = \frac{1}{4\pi} \int_S (\mathbf{K} \times \nabla) \frac{e^{-jkr}}{r} dS \quad (1b)$$

The coordinate system is shown in Fig. 1.

Since these equations (Ref. 6) are exact, the scattering problem consists of (1) determining the currents  $\mathbf{K}$  and (2) evaluating the integrals.

One method for obtaining the currents is to use the following boundary condition on  $S$ :

$$\mathbf{n} \times (\mathbf{E}_i + \mathbf{E}_s) = 0 \quad (2)$$

This method leads to the following integral equation for the currents  $\mathbf{K}$ :

$$-\mathbf{n} \times \mathbf{E}_i = -\mathbf{n} \times \frac{j}{4\pi\omega\epsilon} \int_S [(\mathbf{K} \cdot \nabla) \nabla + k^2 \mathbf{K}] \frac{e^{-jkr}}{r} dS \quad (3)$$

This formulation has been known for some time (Ref. 6), but the speed and size of the present generation of computers have made the technique much more

attractive. This integral equation method has been used recently with great success in solving two-dimensional scattering problems and problems involving a surface of revolution (Ref. 7). The major disadvantage of this approach is that, in these restricted cases, the surface must not be wider than approximately 20 wavelengths (Refs. 8 and 9). In the case of an arbitrary surface, the limitation is more severe.

An approximate alternate method for obtaining the currents is to assume that, on areas directly illuminated by the source, the currents are the same as they would be if the incident fields were reflected optically:

$$\mathbf{K} = 2\mathbf{n} \times \mathbf{H}_i \quad (4)$$

On shadowed regions, the currents are assumed to be zero. These assumptions are known as the physical-optics approximation.

For smooth reflectors, this approximation is valid in the limit of zero wavelength (in Ref. 10, Cullen gives proof of this for the case of a convex body illuminated by a plane wave) and, in contrast to the integral-equation method, the approximation improves as surface dimensions become large compared to a wavelength.

However, not only does this approximation yield results when it is not practical to apply the integral-equation technique, but it yields the results almost free in terms of computer time. Therefore, the generally time-consuming integral-equation technique should be used only when the physical-optics approximation is inadequate. In the typical scattering problem, the currents are of little direct interest; they are only an intermediate step toward obtaining the scattered fields. Therefore, the physical-optics approximation should not be judged by whether it yields the correct currents, but by whether the final computed scattered fields are correct.

In some cases Eq. (4) is a poor approximation of the true currents, but it results in an excellent approximation of the true scattered fields. The reason for this may be understood by the following: let the true currents be  $\mathbf{K}$ , and define  $\Delta\mathbf{K}$  by

$$\Delta\mathbf{K} \equiv \mathbf{K} - 2\mathbf{n} \times \mathbf{H}_i \quad (5)$$

Typically,  $\Delta\mathbf{K}$  exhibits oscillatory behavior over the surface, and its net integrated contribution to the scattered fields is small. Watson (Ref. 11) has shown this analytically for the particular case of the fields in the focal region of a paraboloid that is large with respect to a

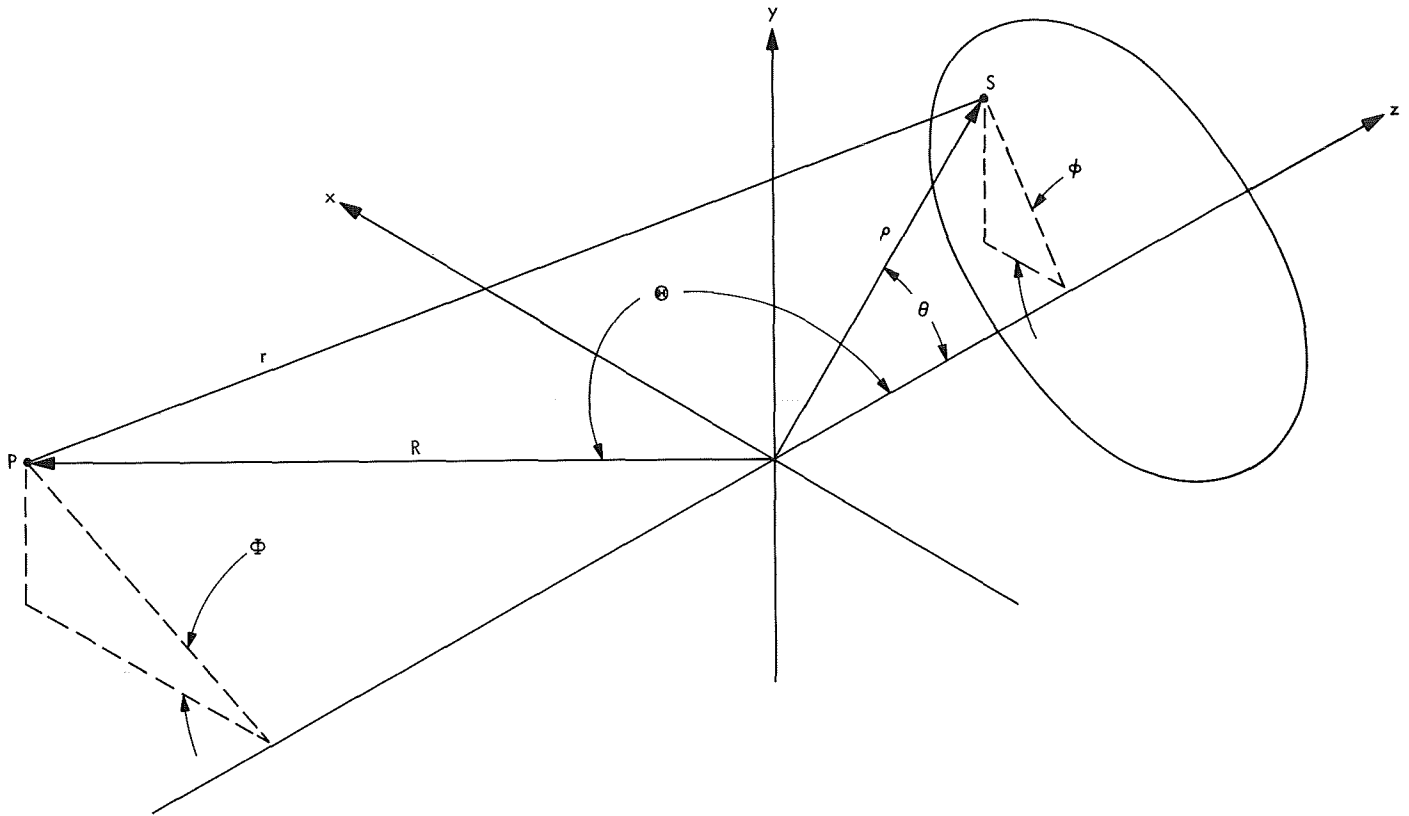


Fig. 1. Coordinate system

wavelength. Presumably, the primary effect of  $\Delta\mathbf{K}$  is to produce local fields that represent stored rather than propagating energy.

The similarity between the currents computed by the integral equation technique and the physical-optics approximation has also been pointed out by Andreassen. The oscillatory nature of the difference may be seen in his results that are for scattering surfaces only a few wavelengths wide (Ref. 9). Surface currents for small spheres—0.18 to 3.2 wavelengths radius—obtained from a classical boundary value solution are given by King and Wu in Ref. 12, and even here, Eq. (4) is a reasonable approximation.

Therefore, Eq. (4) is not only a *necessary* step toward reaching a result in problems involving a large reflector, but also a sound engineering approximation for surfaces larger than a few wavelengths in size.

If it is assumed that  $\mathbf{n}$  and  $\mathbf{H}_i$  are known, the integrals of Eq. (1) can be evaluated to obtain the scattered fields at any point. However, the problem is greatly reduced if it is assumed that  $\mathbf{E}_s$  and  $\mathbf{H}_s$  are to be evaluated only at

very large distances from S. As  $R \rightarrow \infty$ ,  $\mathbf{E}_s$  and  $\mathbf{H}_s$  must satisfy the far-field relations:

$$\mathbf{H} = \left( \frac{\epsilon}{\mu} \right)^{1/2} \mathbf{i}_R \times \mathbf{E} \quad (6a)$$

and

$$\mathbf{E}(R, \theta, \phi) = [E_\theta(\theta, \phi) \mathbf{i}_\theta + E_\phi(\theta, \phi) \mathbf{i}_\phi] \frac{e^{-jkR}}{R} \quad (6b)$$

Therefore, only  $E_\theta$  and  $E_\phi$  need be evaluated. Also in this case, Eq. (1) may be written so that the only term in the integrand that is dependent on the location of the field point P is the argument of the exponential term. The remainder of the integrand may be initially computed over S and stored to be used repeatedly in the evaluation of the integral for every field point P. This method represents a substantial reduction in computer time.

The assumption that the scattered fields are to be evaluated at  $R = \infty$  does not represent a loss of generality. If near-field results are required, they may be

obtained from the far-field data using the spherical-wave expansions described later.

The far-field version of Eq. (1), with the currents that are obtained from Eq. (14), is derived in Silver (Ref. 6):

$$\mathbf{E}_s(P) = I_0 \mathbf{i}_0 + I_\Phi \mathbf{i}_\Phi \frac{e^{-jkR}}{R} \quad (7)$$

where

$$\mathbf{I} = -\frac{j\omega\mu}{2\pi} \int_S (\mathbf{n} \times \mathbf{H}_i) \exp(jk\mathbf{p} \cdot \mathbf{i}_R) dS$$

The final result, the total field  $\mathbf{E}_T$ , is obtained by adding the incident fields to the scattered fields as shown in Eq. (8):

$$\mathbf{E}_T = \mathbf{E}_s + \mathbf{E}_i \quad (8)$$

The problem has now been reduced to the following steps: (1) specify the surface data  $\rho$  and  $\mathbf{n}$ , (2) specify the incident field data  $\mathbf{H}_i$ , and (3) evaluate the integral.

### III. Specification of the Scattering Surface

In the derivation of the scattering integral equations (see Eq. 1), the surface  $S$  is assumed to completely enclose some region of space. Examples of surfaces are a sphere (which separates space into an infinite region and a finite region) and an infinite plane (which separates space into two infinite regions). There are obvious numerical difficulties with infinite surfaces; the finite closed surface also presents some potential problems which are discussed in the following paragraphs.

As discussed in Section IV, the origin of the coordinate system is determined by the location of the source, and the natural system in which to describe the incident fields is spherical coordinates. To define a surface, it is necessary and sufficient to specify one variable as a function of the other two. The most natural and convenient choice for this problem is a function  $\rho(\theta, \phi)$ . In the case of a sphere that does not enclose the origin, a line  $\theta = \text{const}$   $\phi = \text{const}$ , in general, intersects the surface twice so it cannot be described by a single function of this form. However, with the assumed current approximation, the back portion of the sphere has zero currents and can be ignored. The illuminated portion is precisely the portion that can be defined by a single function. There is still one potential difficulty. At the point at

which  $\mathbf{p}$  is just tangent to the surface, the partial derivative  $\partial\rho/\partial\theta$  is infinite and, as will be seen shortly, is required. Therefore, it is necessary to discard a further portion of the surface. In the case of an infinite surface, all but some finite portion is ignored. In both cases the remainder may be identified as a truncated surface. Therefore, an "arbitrary reflecting surface" is defined as a surface that can be represented by a function  $\rho(\theta, \phi)$  whose partial derivatives exist at every point on the surface. This definition is basically a consequence of the physical-optics approximation used to obtain the currents, and is sufficiently general to include virtually all antenna reflectors. (Discontinuous surfaces may be represented by two or more segments satisfying these requirements, and the results may be superimposed.)

Because this surface may be thought of as a portion of a closed surface, the derivation of Eq. (1) is still valid, except for one point. A truncated surface has an edge, and the currents obtained from Eq. (14) will in general be discontinuous there. To ensure that the results are consistent with Maxwell's equations and the radiation condition, Silver (see Ref. 6) introduced a line charge on this edge in the derivation of Eq. (7). Sancer (Ref. 13) has shown that the surface integrals of Eq. (1) intrinsically contain the contour integral representing the contribution of this charge, and that the term introduced by Silver arises automatically as a mathematical consequence of truncating the surface.

The surface data required to evaluate the integral of Eq. (7) are  $\rho$  and  $\mathbf{n}$ . Using a result from differential geometry (Ref. 14), Dickinson (Ref. 15) pointed out the useful relationship:

$$\mathbf{n} dS = \frac{\partial \mathbf{p}}{\partial \phi} \times \frac{\partial \mathbf{p}}{\partial \theta} d\theta d\phi \quad (9)$$

where

$$\left. \begin{aligned} \rho &= \rho \mathbf{i}_\rho \\ \frac{\partial \rho}{\partial \phi} &= \frac{\partial \rho}{\partial \theta} \mathbf{i}_\rho + \rho \sin \theta \mathbf{i}_\phi \\ \frac{\partial \rho}{\partial \theta} &= \frac{\partial \rho}{\partial \theta} \mathbf{i}_\rho + \rho \mathbf{i}_\theta \end{aligned} \right\} \quad (10)$$

Therefore, specification of  $\rho$  and its partial derivatives is a complete description of the surface. Since these quantities must be specified on a set of points on  $S$ , which will be called the integration grid and can easily involve

several thousand points, the storage and transmission of a description of a surface in this form can be a problem. For this reason, it is useful to achieve some sort of data compression. One trivial case is that in which the surface is completely described by an analytic expression; for example, a paraboloid whose focus is at the origin, is completely described by a single number  $f$  as shown in Eq. (11):

$$\rho = f \sec^2 \frac{\theta}{2} \quad (11)$$

A compromise between the complete generality of two-dimensional tabular input and the severe restriction of a simple analytic equation may be developed from the general form:

$$\rho(\theta, \phi) = \sum_{m=0}^M a_m(\theta) \cos m\phi + b_m(\theta) \sin m\phi \quad (12)$$

Then

$$\frac{\partial \rho}{\partial \theta} = \sum_{m=0}^M \frac{\partial a_m}{\partial \theta} \cos m\phi + \frac{\partial b_m}{\partial \theta} \sin m\phi \quad (13a)$$

and

$$\frac{\partial \rho}{\partial \phi} = \sum_{m=1}^M -ma_m(\theta) \sin m\phi + mb_m(\theta) \cos m\phi \quad (13b)$$

Suppose that for a given surface it is necessary to include terms of order up to  $M$  to determine  $\rho$  with sufficient accuracy; then by the sampling theorem it would be necessary to specify  $\rho$  at  $2M$  values of  $\phi$  to accurately represent details of the surface variations using tabular data. Therefore, this Fourier type of representation is in principle always at least as efficient as an ordinary tabular representation.

If  $M$  is allowed to be arbitrarily large, Eq. (12) is in fact completely general; however, the reason for choosing this form is that the first few terms are sufficient to accurately represent a slightly tilted surface of revolution, which is a case of substantial practical importance. For example, Fig. 2 shows the peak error that occurs when a tilted plane reflector is represented by three terms of the expansion,  $a_0(\theta)$ ,  $a_1(\theta)$ , and  $a_2(\theta)$ . (The axis of rotation is taken to be the  $y$ -axis, so all of the  $b_m(\theta)$  terms are zero.) This error is in general a function of the curvature of the reflector, but the data shown in the figure are

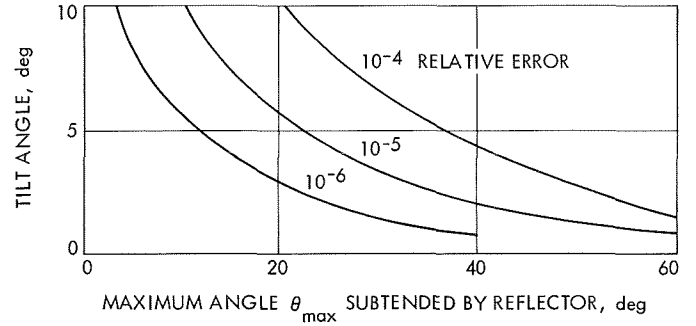


Fig. 2. Peak error with three Fourier components used to represent a tilted reflector

representative of the accuracy that may be obtained with this method. A reasonable *a posteriori* estimate of accuracy may be obtained by extrapolating the value of the first neglected term in the expansion. For example, for a tilt angle of 1 deg, the coefficients at  $\theta_{max} = 20$  deg are  $a_0 = 1.064199$ ,  $a_1 = 0.006761$ , and  $a_2 = 0.000021$ . The values decrease with a nearly constant ratio, and a reasonable estimate for the neglected third-order term is about  $10^{-5}$  to  $10^{-6}$ , which is approximately the peak error. The relatively poor accuracy obtained for a tilt angle of 10 deg at  $\theta_{max} = 40$  deg is also predictable from the coefficients ( $a_0 = 1.320015$ ,  $a_1 = 0.197465$ , and  $a_2 = 0.014608$ ) and would lead to an expected relative error of approximately  $10^{-3}$ .

The derivatives of the  $a_m(\theta)$  terms must also be written in as tabular functions of  $\theta$ ; therefore, six input functions are required in this example where  $M=3$  and all  $b_m(\theta)$  terms are identically zero. The number of  $\phi$  values at which  $\rho$  must be known is determined by the requirements of the numerical integration technique and is usually quite large (from fifty to several hundred values is typical). Therefore, the specification of only six functions is a substantial reduction in the quantity of data required. A computer program to obtain these functions for the case of an arbitrary tilt and translation of a surface of revolution is described in Appendix C.

#### IV. Specification of the Incident Fields

The most common method (Refs. 1, 6, and 16) for obtaining the magnetic field on the surface  $S$  is to assume that the incident fields satisfy the far-field relations:

$$\mathbf{H}_i = \left( \frac{\epsilon}{\mu} \right)^{1/2} \mathbf{i}_\rho \times \mathbf{E}_i \quad (14a)$$

and

$$\mathbf{E}_i(\rho, \theta, \phi) = [E_\theta(\theta, \phi) \mathbf{i}_\theta + E_\phi(\theta, \phi) \mathbf{i}_\phi] \frac{e^{-jk\rho}}{\rho} \quad (14b)$$

Not only is this a relatively simple form, but  $E_\theta(\theta, \phi)$  and  $E_\phi(\theta, \phi)$  correspond to the quantities conventionally measured on an antenna pattern range, so experimental data may be easily used as input for the incident fields. Also, as discussed in Section II, theoretical patterns are easier to compute in this form.

Equation (14) represents a spherical wave diverging from the origin; therefore, it is not valid unless the origin is located at the source phase center. The phase pattern about this point should be constant, but in practice the relative phase may vary by 90 deg or more over the region of interest. A nonconstant phase pattern may be taken into consideration if  $E_\theta$  and  $E_\phi$  are allowed to be complex valued, but the radial components of the field implied by the nonconstant phase pattern are usually assumed negligible.

The use of this assumption provides good results for many practical problems. Serious errors can arise, however, when the surface is "close" to the source. This fact has been recognized by Zucker and Ierley, who directly computed the near-field radiation of a conical horn in evaluating the scattered pattern of a near-field Cassegrainian subreflector (Ref. 3). An alternative approach is to take experimental data at range values corresponding to the location of the surface; this was done by Hogg and Semplak (Ref. 17).

The disadvantage of the first method is that horn patterns computed in this way frequently do not agree with experimental data, since it is necessary that strong assumptions be made about the fields in the aperture of the horn. The disadvantage of the second method is that each different subreflector configuration in principle requires a new set of experimental data. These problems are obviated by the use of the spherical-wave expansions that are discussed later in this section.

A graphic demonstration of the error that may be introduced if far-field behavior is assumed for the incident fields is provided by the case of scattering from a large plane reflector. In the case of an infinite plane, the result must be the reflection of the incident pattern, independent of frequency. Also, in the case of an infinite plane, Eq. (4) is valid, independent of frequency; therefore, this is an excellent direct test of Eq. (14).

The cases shown in Fig. 3 (and one additional case at  $z = 160 \lambda$ , which is not illustrated) were evaluated using the far-field form for the incident fields given by Eq. (14). The fields were derived from the experimental pattern of a conical horn with an aperture diameter of  $4.671 \lambda$ , illustrated at the origin of Fig. 3. The actual numerical integration was truncated at an angle at which contributions became negligible. The disks shown in Fig. 3 represent the region over which the integral was actually evaluated. It should be noted that horn-reflector interactions are *not* considered, so this is not intended to be a perfect duplication of a real experimental situation, but rather a numerical test of the far-field assumption given by Eq. (14).

The computed scattered fields based on the far-field approximation of the incident field for the six cases are

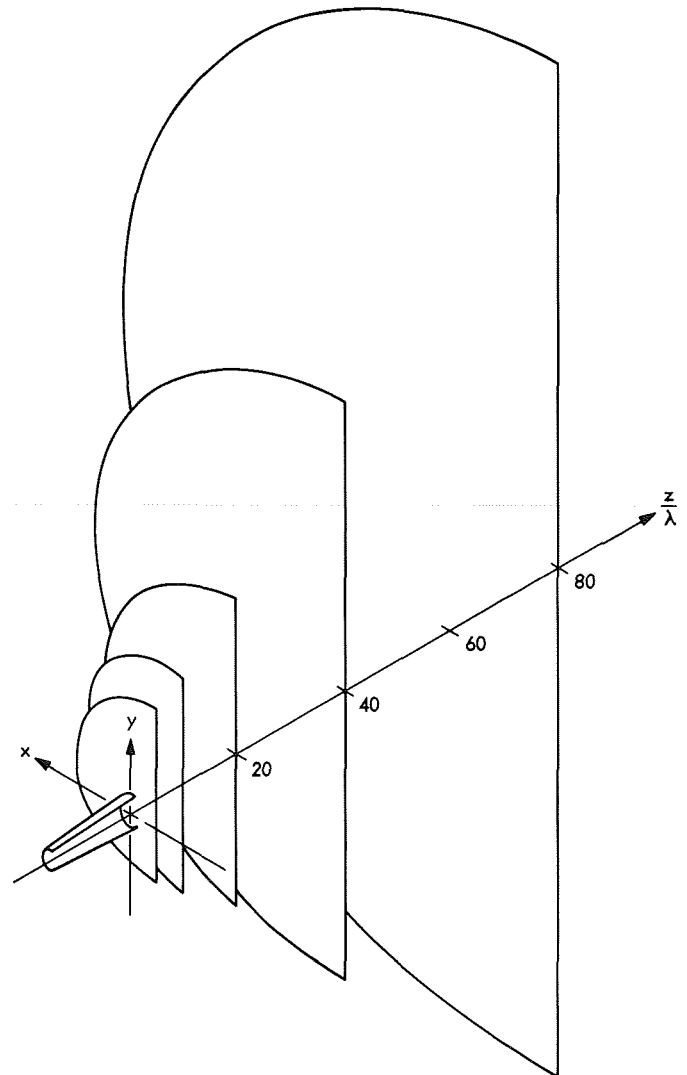


Fig. 3. Plane reflector cases

shown in Fig. 4 along with the incident-field pattern. Only the  $\Phi$ -component is illustrated, but the behavior of the  $\Theta$ -component is similar. All of these patterns should be identical, but the discrepancies are large. In addition to the severe distortion of the reflected main lobe when  $z = 5$  and  $10 \lambda$ , a spurious back lobe appears in all cases—that is, the plane that was assumed to be a perfect reflector behaves as if it were partially transparent.

In the special case of a plane reflector, the scattered fields  $\mathbf{E}_s$  are identical in the reflected and backward directions. When the incident field is added (Eq. 8), it should cancel this backward radiation. Therefore, this

back lobe is identically the error between the reflected pattern and the incident pattern. (The error includes both amplitude and phase differences since it is the difference between complex-valued quantities.)

The power in the computed scattered patterns has been calculated as a percentage of the power in the incident fields and is shown in Fig. 5. As indicated in Fig. 5, the main (reflected) lobe alone accounts for 100 percent of the incident power in all cases, so the total power in the computed pattern exceeds the incident power. This excess is, of course, physically impossible. The spurious power in the back lobe is strongly dependent on the distance

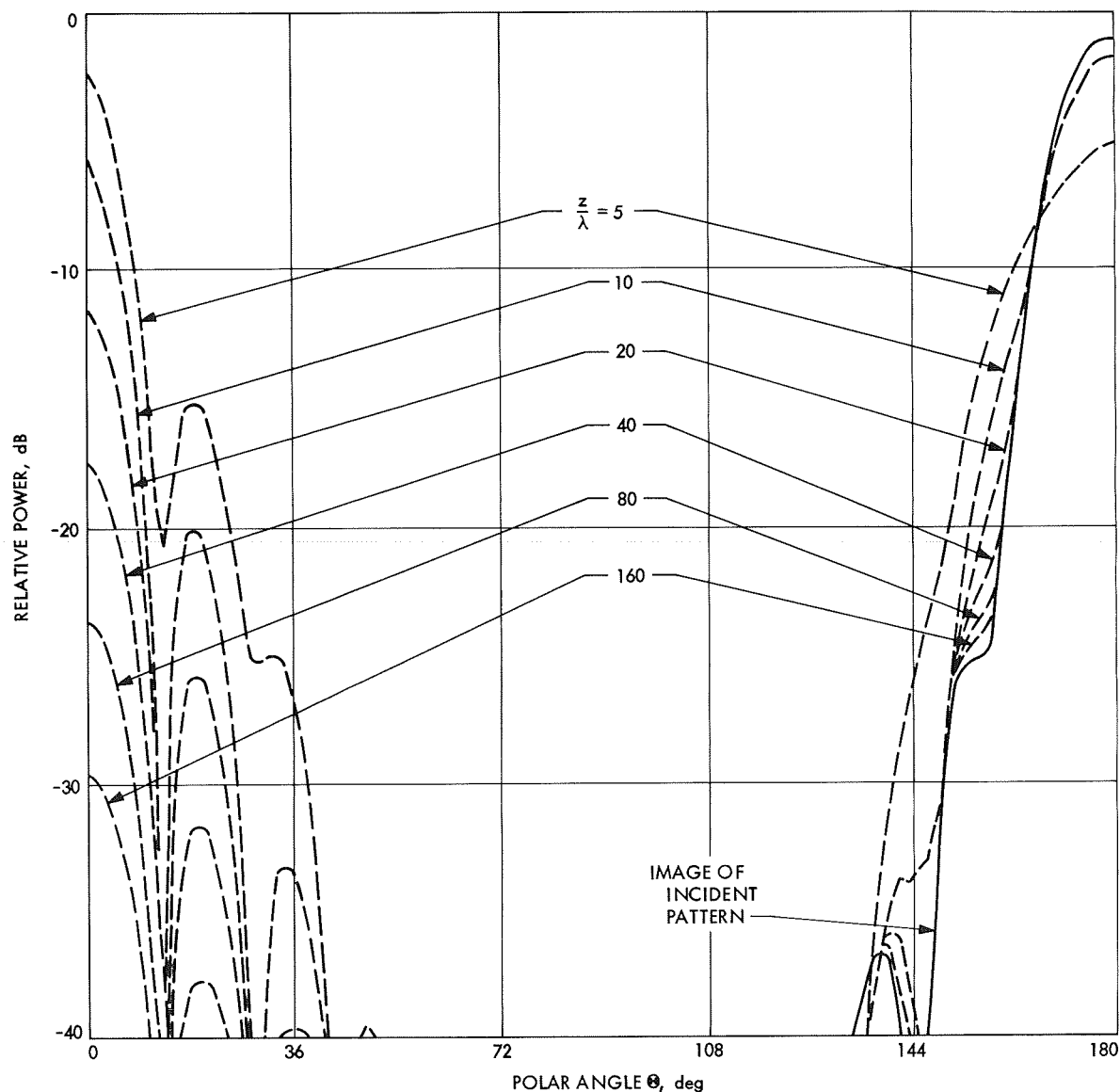
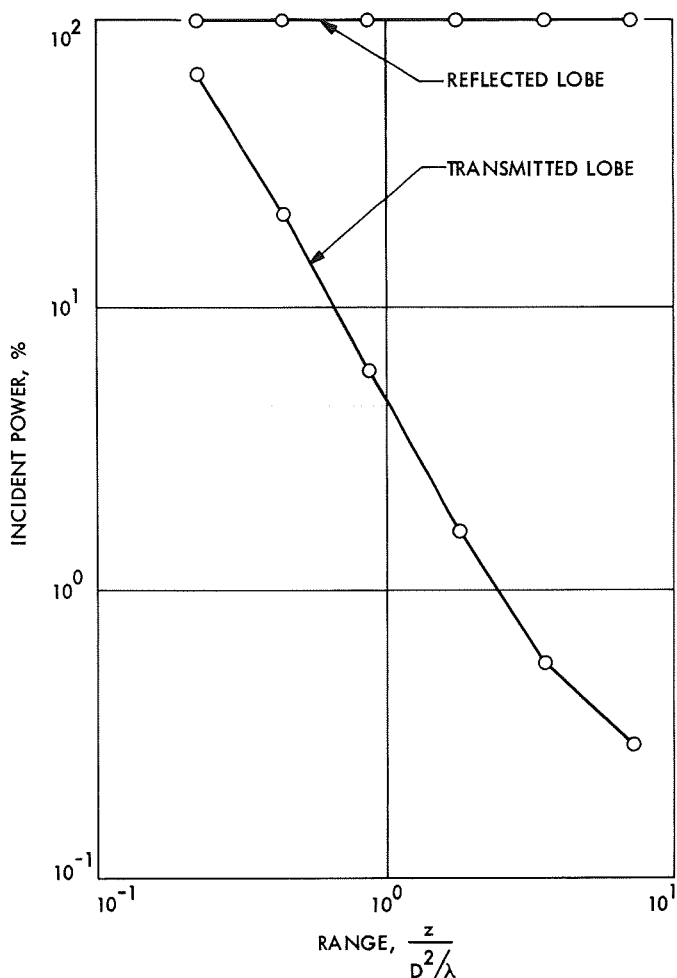


Fig. 4. Computed amplitude patterns for plane reflector cases; far-field assumption for incident fields





from the source to the surface, which is plotted in units of  $D^2/\lambda$  where  $D$  is the diameter of the source. At  $z=2D^2/\lambda$ , the traditional far-field boundary line, this power has dropped to less than 2 percent.

The behavior of the phase pattern of  $E_*$  is shown in Fig. 6. These phase data have been transformed to a new coordinate system; the origin was moved from the point at the source phase center to the image of this point behind the plane reflector where the phase center of the total scattered fields should be located. In fact, this is very nearly the phase center for the first three cases; the patterns of the last cases are those of a source whose phase center is closer to the plane surface.

Therefore, instead of the total scattered fields appearing to arise from a perfect image of the source, in the three worst cases the apparent source is distorted and has moved toward the plane. In most practical situations, the agreement shown for the three best cases would be

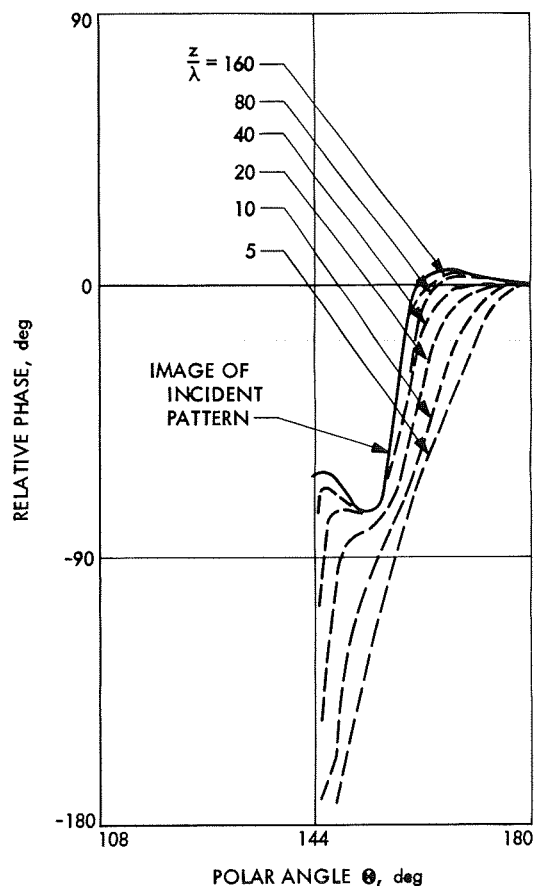
adequate, but it is significant that even at the furthest distance  $z = 160 \lambda$  ( $7D^2/\lambda$  of the feed) there is noticeable error in the results.

This error may be eliminated entirely by the use of a representation of the incident fields that is valid in the near field. Spherical waves are a well known set of solutions to Maxwell's equations that satisfy this requirement. If the incident fields satisfy Maxwell's equations, then coefficients  $a_n$  and  $b_n$  may be found such that

$$\mathbf{H}_i(\rho, \theta, \phi) = \frac{k}{j\omega\mu} \sum_{n=0}^{\infty} a_n \mathbf{n}_n + b_n \mathbf{m}_n \quad (15)$$

where  $\mathbf{n}_n$  is a transverse electric (TE) and  $\mathbf{m}_n$  is a transverse magnetic (TM) spherical wave, as given by Eq. (A-1) in Appendix A.

The method for obtaining coefficients so that the wave expansion will match an arbitrary input pattern in the far field is also discussed in Appendix A; therefore, it will simply be assumed that coefficients have been obtained



**Fig. 6. Computed phase patterns for plane reflector cases; far-field assumption for incident fields**

to describe the incident field. The form of Eq. (15) completely eliminates the approximation of Eq. (14) by directly yielding  $\mathbf{H}_i$  on the surface. However, the evaluation of the spherical-wave expansion does involve computer time. Although this time is typically less than 10 percent of the total, the comment made earlier with regard to the integral-equation technique for obtaining the currents also applies here: the expansion should be used only when the far-field form is inadequate.

Clearly, the far-field form is inadequate in cases  $z = 5$ , 10, and  $20 \lambda$  of the data presented in Figs. 4 and 6. These cases have been recomputed using a spherical-wave coefficient specification of the incident fields. The reflected patterns are virtually indistinguishable from the incident pattern, as shown in Fig. 7. Phase data, transformed to the image point, are shown in Fig. 8; clearly, the error has been eliminated.

Now that it has been established that the far-field assumption yields incorrect results that are corrected when a spherical-wave expansion is used, it is of interest to consider the physical mechanism of the discrepancy. Since the theory is based on current integration, the surface current distribution must reflect the essential difference between the two methods. The amplitude and phase of the  $y$  component (the principal polarization) of the surface currents at  $\phi = 0$  are compared for three cases in Figs. 9 and 10. The currents in Figs. 9 and 10 are plotted versus the Cartesian coordinate  $x$ , rather than the polar angle  $\theta$  (see Figs. 1 and 3), to illustrate the true relative spatial distributions of the three cases.

When the far-field approximation is assumed, the diameter of the region containing significant current magnitudes linearly approaches zero as the source approaches the reflector. Typically, a smaller source will result in a broader radiation pattern, but in this case the phase of the surface currents becomes more uniform as the source becomes smaller and this fact tends to make the pattern narrower. Over a considerable range of  $z$  values, these effects nearly cancel out, as demonstrated by the resulting radiation patterns shown in Fig. 4. However, in the limiting case of a point source, a broad dipole pattern would be expected with the phase center of the pattern on the reflecting surface. This is in fact exactly the trend exhibited by the data for the far-field case in Figs. 4 and 6. As mentioned previously, the surface currents radiate symmetrically on both sides of the plane, and because a broader pattern cannot result in perfect destructive interference with the incident pattern, the result is the back lobe which appears in Fig. 4.

When the spherical-wave expansion is used, the phase behavior is very similar to the far-field result, except that the 180-deg discontinuities are smoothed out. (Since the phase data are modulo 360 deg, these discontinuities could be in either the positive or negative direction. They were drawn to match the spherical-wave phase data as closely as possible.) The major difference between the spherical-wave and far-field cases is that the patterns of the current magnitudes are broadened, particularly in the  $z = 5 \lambda$  case. Because the phase patterns are nearly the same, the larger source produces a narrower radiation pattern that brings the result into agreement with the incident pattern, as shown in Fig. 7.

It is of interest to note that the radial field component of the incident magnetic field—which is completely neglected in the far-field approximation—makes a noticeable contribution to the total current, as shown in Fig. 11. When  $z = 5 \lambda$ , this current contributes about 3 percent of the field strength of the scattered pattern on axis, so a small but not negligible part of the correction to the far-field approximation is the inclusion of the radial field components. However, these surface current data lead to the conclusion that the primary cause of the error resulting from the use of the far-field approximation is that the current magnitude patterns are too narrow.

The computer program developed in this report (see Appendix D) includes a subroutine FIELDS that provides the main program with values of  $\mathbf{H}_i$  on the surface integration grid. Two such subroutines were written. In one subroutine the incident fields are specified by a set of spherical-wave coefficients  $a_n$  and  $b_n$ . In the other subroutine the far-field relations of Eq. (14) are assumed, and  $E_\theta$  and  $E_\phi$  are represented by exactly the same type of series as the surface data (see Eq. 12). In the case of the field data, the  $m = 1$  azimuthal variation term is of special significance and is the only term present in an important class of sources (Ref. 18). For this reason, the FIELD subroutines were written for any single value  $m$  for the order of azimuthal variation. If more than one value is required, the cases may be run individually and the results superimposed. The far-field subroutine is intended for cases in which the reflecting surface is sufficiently removed from the source, or in which the available number of spherical waves (limited by storage and other considerations) is inadequate to expand the incident-field pattern. The spherical-wave subroutine should be used whenever the reflector is close to the source, and in borderline cases (which include most practical Cassegrainian antenna systems) when assured accuracy is the overriding concern.

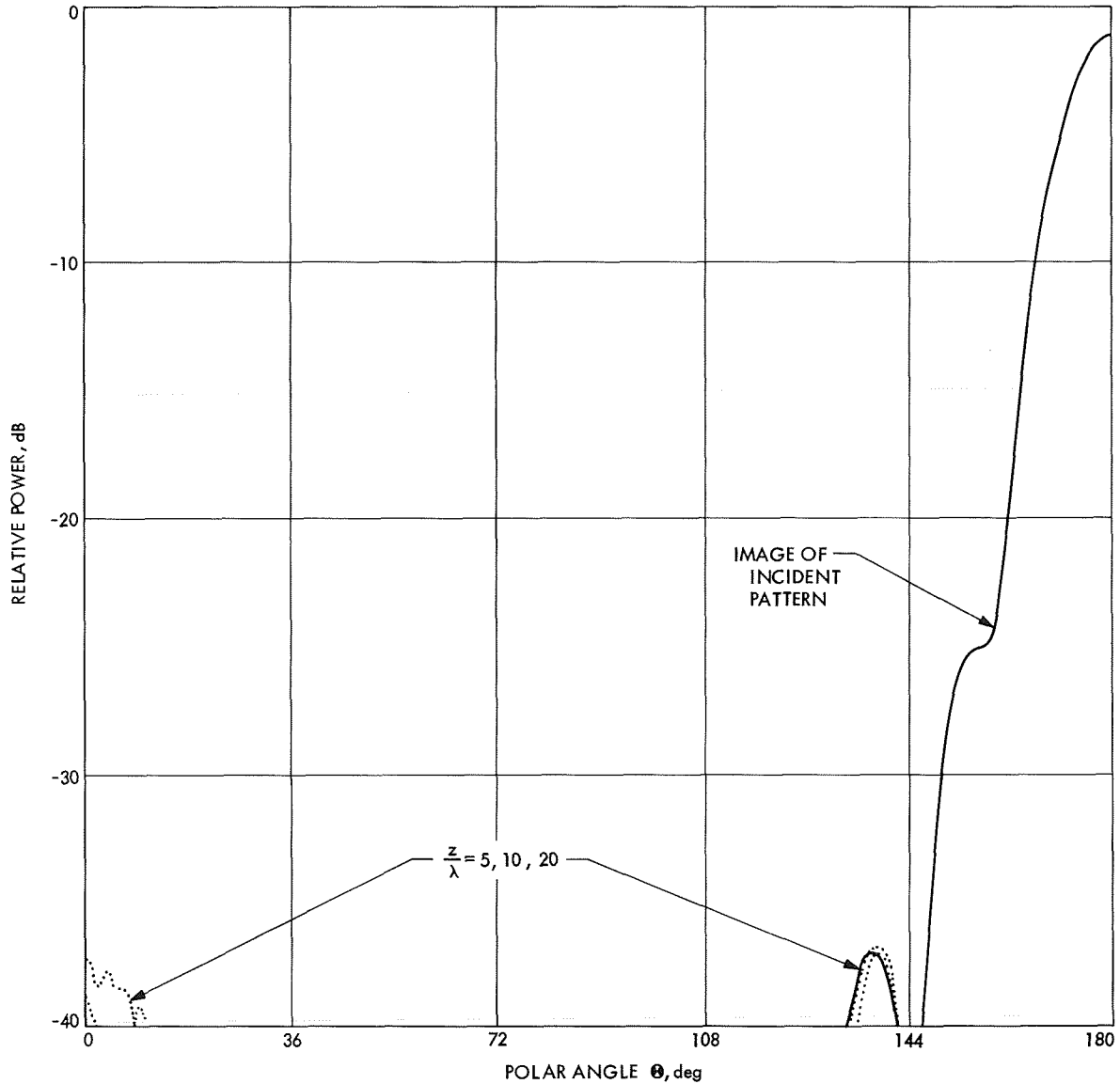


Fig. 7. Computed amplitude patterns for plane reflector cases; spherical-wave expansion of incident fields

## V. Evaluation of the Integral

The numerical evaluation of the integral, Eq. (7), may justifiably be considered the most important part of the scattering problem. A primary engineering function of a scattering program is the evaluation of the effect of one or more parameters on the resulting pattern; this application is severely restricted if each case requires 6 or 8 h of computer time.

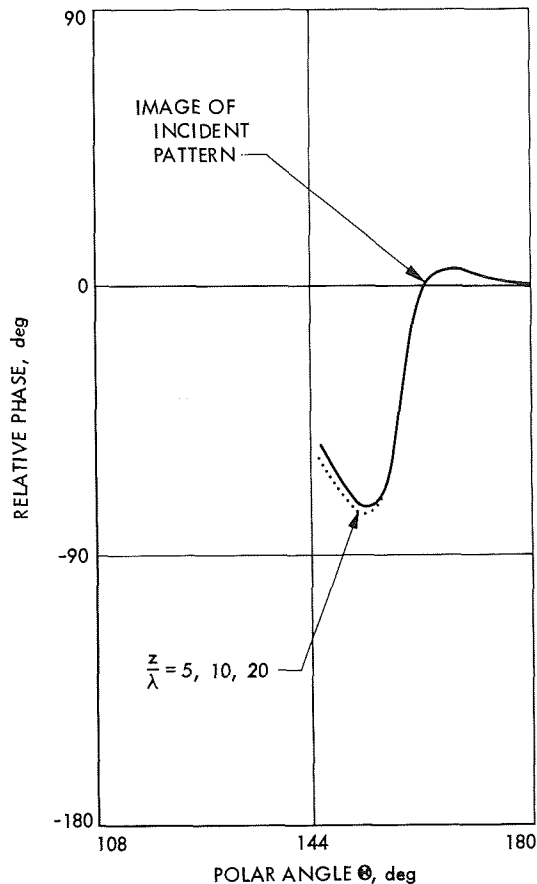
The fast Fourier transform technique has been applied to the evaluation of diffraction integrals by Pratt and Andrews (Ref. 19), but this formulation is applicable only when the argument of the exponential term is linear

in each of the variables of integration. In the case considered here, the integral to be evaluated (see Eq. 7) may be written as follows:

$$\mathbf{I}(\Theta, \Phi) = -\frac{j\omega\mu}{2\pi} \int \int_s \mathbf{F}(\theta, \phi) e^{jk\gamma(\theta, \phi, \Theta, \Phi)} d\theta d\phi \quad (16)$$

where

$$\begin{aligned} \gamma &\equiv \boldsymbol{\rho} \cdot \mathbf{i}_R - \rho \\ &= \rho [\sin \theta \sin \Theta (\cos \phi \cos \Phi + \sin \phi \sin \Phi) \\ &\quad + \cos \theta \cos \Theta - 1] \end{aligned} \quad (17)$$



**Fig. 8. Computed phase patterns for plane reflector cases; spherical-wave expansion of incident fields**

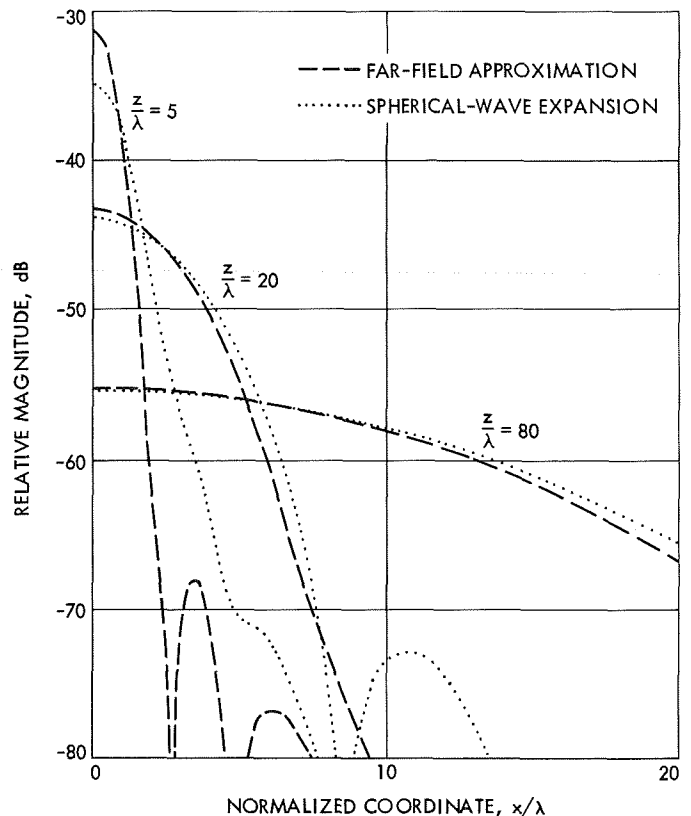
Since  $\gamma$  is a nonlinear function of  $\theta$  and  $\phi$ , the fast Fourier transform technique is not applicable. Techniques that have been successfully applied to Eq. (16) are Gaussian quadrature, which was used by Allen (Ref. 20), and Romberg quadrature, which was recently used by Rusch and Strachman (Ref. 21). The Romberg method is particularly useful in the case of computing the main-beam and near-sidelobe regions of a high-gain pattern, and could in fact be used with the method presented below to increase efficiency in this situation.

Before describing the integration technique developed by the author, some general features involved in evaluating this integral will be discussed. Suppose  $I(\theta, \phi)$  is to be evaluated on a set of points (the output grid) given by  $(\theta_j, \phi_k)$ ,  $1 \leq j \leq JMAX$ ,  $1 \leq k \leq KMAX$ . Typical values are  $JMAX = 91$ ,  $KMAX = 2$ . The integral will be evaluated numerically when the integrand is specified at a set of points (the integration grid) given by  $(\theta_m, \phi_n)$ ,  $1 \leq m \leq MMAX$ ,  $1 \leq n \leq NMAX$ . A modest grid size is  $MMAX = 50$ ,  $NMAX = 181$ . (The effect of these grid

parameters on computed pattern accuracy is discussed below and is considered in detail in Ref. 22.)

As mentioned in Section II, with the far-field assumption,  $\mathbf{F}$  does not depend on  $\theta$  and  $\phi$ , so it need only be computed once on the integration grid—9050 points in this example. This operation requires little computer time, but since  $\mathbf{F}$  has three Cartesian components (each of which is complex),  $6 \times 9050 = 54,300$  values must be stored. Many computers do not have this capacity. Therefore, the program was written so that several integration grids may be specified—i.e., the reflector is divided into segments—and the resulting scattered fields from each segment are superimposed.

Unavoidably, the function  $e^{jk\gamma}$  must be computed at  $JMAX \times KMAX \times MMAX \times NMAX$  points (1,647,100 in this example). This example demonstrates the tremendous advantage of storing  $\mathbf{F}$  rather than recomputing it for every output point. Also, the number of points (1,647,100) represents the number of summations involved to evaluate the integral for all of the output points. Several steps were



**Fig. 9. Surface currents on plane reflectors; amplitude behavior**

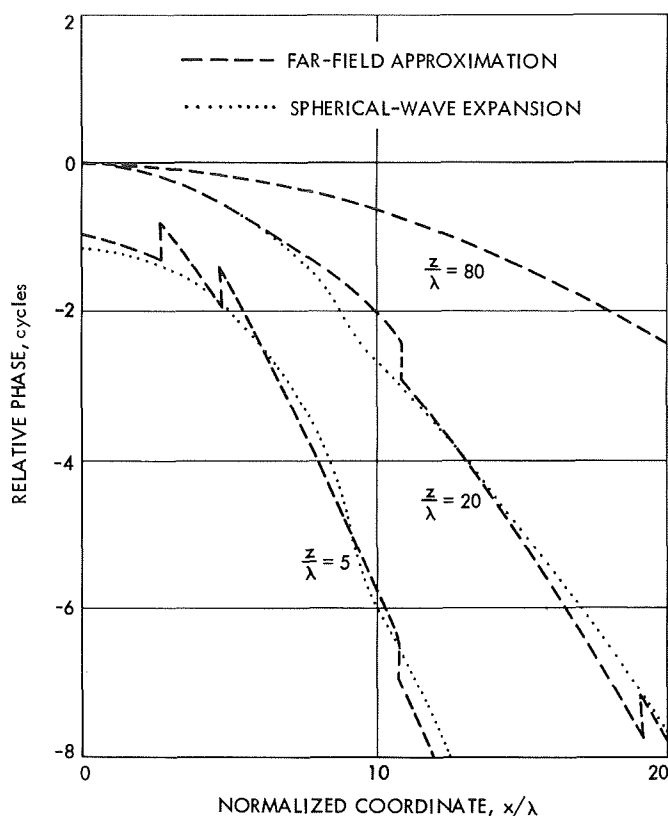


Fig. 10. Surface currents on plane reflectors; phase behavior

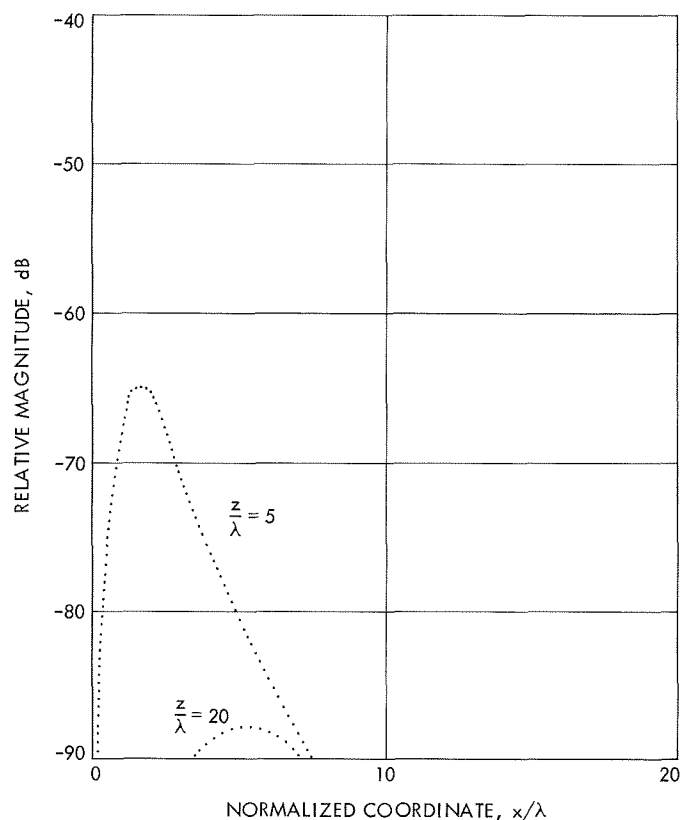


Fig. 11. Surface currents on plane reflectors; contribution of the radial component of the incident fields

taken to minimize the time used in this portion of the program:

- (1) The trigonometric functions  $\sin \theta$ ,  $\cos \theta$ ,  $\sin \phi$ ,  $\cos \phi$ ,  $\sin \Theta$ ,  $\cos \Theta$ ,  $\sin \Phi$ ,  $\cos \Phi$  are precomputed at the required grid angles and stored so that the computation of  $\gamma$  will involve only multiplication and addition of stored quantities, rather than the time-consuming evaluation of trigonometric functions. For this reason, Eq. (17) is not written in the form  $\cos \phi \cos \Phi + \sin \phi \sin \Phi = \cos (\phi - \Phi)$ , which would have to be computed at every permutation of  $\phi$  and  $\Phi$  values.
- (2) A fast machine-language subroutine was written to compute  $\cos k\gamma + j \sin k\gamma$ . This subroutine is described in Appendix B.
- (3) A special numerical integration technique was developed that reduces the required values of MMAX and NMAX.

The numerical integration technique is described in Ref. 5, but for completeness the description will be partially repeated here. For a fixed output point  $(\Theta_p, \Phi_q)$  the

integral is of the form

$$\int_{\theta_1}^{\theta_M} \int_{\phi_1}^{\phi_N} \mathbf{F}(\theta, \phi) \exp jk\gamma_{pq}(\theta, \phi) d\theta d\phi \quad (18)$$

where

$$\begin{aligned} \mathbf{F} = & \left( \frac{\partial \rho}{\partial \phi} H_\rho + \rho \sin \theta H_\phi \right) \mathbf{i}_\theta \\ & + \left( -\rho \sin \theta H_\theta - \frac{\partial \rho}{\partial \theta} H_\rho \right) \mathbf{i}_\phi \\ & + \left( \frac{\partial \rho}{\partial \theta} \sin \theta H_\phi - \frac{\partial \rho}{\partial \phi} H_\theta \right) \mathbf{i}_\rho \end{aligned} \quad (19a)$$

$$H_\rho \mathbf{i}_\rho + H_\theta \mathbf{i}_\theta + H_\phi \mathbf{i}_\phi \equiv \rho e^{jk\rho} \mathbf{H}_i \quad (19b)$$

Since the unit vectors of Eq. (19) vary over the region of integration, Cartesian components are used in the actual evaluation of the integral.

In the near field, the radial variation of  $\mathbf{H}_i$  will approximate  $e^{-jk\rho}/\rho$  behavior; therefore, if this dominant variation

is factored out,  $H_\rho$ ,  $H_\theta$ , and  $H_\phi$  as defined in Eq. (19) will be nearly constant with respect to  $\rho$ , which makes it much easier to numerically approximate  $\mathbf{F}$ . The  $e^{-jk\rho}$  factor has been included in the path length function  $\gamma$ , and the  $1/\rho$  factor is canceled by a  $\rho$  factor in the term for incremental area  $dS$ . Deviations from  $e^{-jk\rho}$  phase behavior are included by allowing  $H_\rho$ ,  $H_\theta$ , and  $H_\phi$  to be complex valued. (Since a radial component of  $\mathbf{H}_i$  is included, this does *not* involve any assumptions or approximations).

Consider the behavior of the integrand over an incremental area of  $S$ ,

$$\Delta S_{mn} = \{(\theta, \phi) : \theta_m \leq \theta \leq \theta_{m+1}, \phi_n \leq \phi \leq \phi_{n+1}\} \quad (20)$$

Suppose that the physical dimensions of  $\Delta S_{mn}$  are on the order of a wavelength,  $\lambda = 2\pi/k$ . Then the path length term  $jk\gamma$  cannot vary by more than  $2\pi$ , and since electromagnetic fields cannot generally change abruptly over

distances on the order of a wavelength,  $\mathbf{F}$  will not vary much. Thus, in virtually any problem,  $\mathbf{F}$  and  $\gamma$  will be very well behaved and slowly varying over  $\Delta S_{mn}$ . However, the possible  $2\pi$  variation in the exponential term could cause the real and imaginary parts of the integrand to behave like a full cycle of a sinusoid. To apply a technique such as Simpson's rule to the entire integrand would require a further subdivision of  $\Delta S$ . However, if the functions  $\mathbf{F}$  and  $\gamma$  are approximated individually a simple linear form will operate satisfactorily over  $\Delta S_{mn}$ . Explicitly, write

$$\mathbf{F}(\theta, \phi) \simeq \mathbf{a}_{mn} + \mathbf{b}_{mn}(\theta - \theta_m) + \mathbf{c}_{mn}(\phi - \phi_n) \quad (21a)$$

$$\gamma(\theta, \phi) \simeq \alpha_{mn} + \beta_{mn}(\theta - \theta_m) + \xi_{mn}(\phi - \phi_n) \quad (21b)$$

for  $(\theta, \phi) \in \Delta S_{mn}$ .

The method used for determining the coefficients is to best-fit a plane (least squares sense) to the values of the function at the corners of  $\Delta S_{mn}$ . For example, in the case of the function  $\mathbf{F}$  this method yields the following:

$$\mathbf{a}_{mn} = \frac{1}{4} [3\mathbf{F}(\theta_m, \phi_n) - \mathbf{F}(\theta_{m+1}, \phi_{n+1}) + \mathbf{F}(\theta_{m+1}, \phi_n) + \mathbf{F}(\theta_m, \phi_{n+1})] \quad (22a)$$

$$\mathbf{b}_{mn} = \frac{1}{2\Delta\theta_m} [\mathbf{F}(\theta_{m+1}, \phi_n) - \mathbf{F}(\theta_m, \phi_n) + \mathbf{F}(\theta_{m+1}, \phi_{n+1}) - \mathbf{F}(\theta_m, \phi_{n+1})] \quad (22b)$$

$$\mathbf{c}_{mn} = \frac{1}{2\Delta\phi_n} [\mathbf{F}(\theta_m, \phi_{n+1}) - \mathbf{F}(\theta_m, \phi_n) + \mathbf{F}(\theta_{m+1}, \phi_{n+1}) - \mathbf{F}(\theta_{m+1}, \phi_n)] \quad (22c)$$

where

$$\Delta\theta_m = \theta_{m+1} - \theta_m$$

$$\Delta\phi_n = \phi_{n+1} - \phi_n$$

An identical relation holds between the coefficients  $\alpha_{mn}$ ,  $\beta_{mn}$ ,  $\xi_{mn}$  and the function  $\gamma$ . The integration over  $\Delta S_{mn}$  may then be performed analytically, yielding the contribution:

$$\begin{aligned} \Delta \mathbf{I}_{mn} = \exp jk\alpha_{mn} & \left\{ a_{mn} \left[ \frac{\exp jk\beta_{mn}\Delta\theta_m - 1}{jk\beta_{mn}} \right] \left[ \frac{\exp jk\xi_{mn}\Delta\phi_n - 1}{jk\xi_{mn}} \right] \right. \\ & + \mathbf{b}_{mn} \left[ \frac{\Delta\theta_m}{jk\beta_{mn}} \exp jk\beta_{mn}\Delta\theta_m - \left( \frac{\exp jk\beta_{mn}\Delta\theta_m - 1}{(jk\beta_{mn})^2} \right) \right] \left[ \frac{\exp jk\xi_{mn}\Delta\phi_n - 1}{jk\xi_{mn}} \right] \\ & \left. + \mathbf{c}_{mn} \left[ \frac{\exp jk\beta_{mn}\Delta\theta_m - 1}{jk\beta_{mn}} \right] \left[ \frac{\Delta\phi_n}{jk\xi_{mn}} \exp jk\xi_{mn}\Delta\phi_n - \left( \frac{\exp jk\xi_{mn}\Delta\phi_n - 1}{(jk\xi_{mn})^2} \right) \right] \right\} \quad (23) \end{aligned}$$

Since it is possible for  $\beta_{mn}$  or  $\xi_{mn}$  to be near or equal to zero, it is necessary to develop separate equations for this case (they are easily derived from the above equation) to avoid large numerical errors.

It should be noted that many of the computations involved in evaluating each single vector component do not have to be repeated for the remaining five components. For example, in Eq. (23) only the components of the  $\mathbf{a}_{mn}$ ,  $\mathbf{b}_{mn}$  and  $\mathbf{c}_{mn}$  coefficients will change. Therefore, although the six vector components present a severe storage problem, the effect on computer time is considerably less than a factor of six.

The basic idea behind this method—to isolate the oscillatory behavior of the integrand—is conceptually similar to the Eikonal technique used in deriving geometrical optics from Maxwell's equations (Ref. 23) and is also a feature of Filon's method for evaluating Fourier integrals (Ref. 24). In fact, Allen has also applied Filon's method to antenna problems and concluded that it is competitive with Gaussian quadrature (Ref. 20). The essential difference of the method presented here is that the "frequency" terms  $\beta_{mn}$  and  $\xi_{mn}$  are re-estimated for each incremental area of integration, rather than assumed as known constants.

If radiation integrals are considered as analytic forms of a Huygens type of principle in which infinitesimal electric or magnetic dipoles radiate a simple pattern and are summed as in an array, then another way of interpreting this technique is to describe it as replacing infinitesimal elements with elements about a wavelength square that radiate a more complicated pattern as given by Eq. (23).

Although this technique appears obvious and has an intuitive appeal from an engineering standpoint, it is somewhat unusual mathematically. The integral is a linear operator, and virtually all quadrature formulas are also linear, except this one. The easiest way to show this is to note that the functions 1 and  $e^{jk\theta}$  are both of the form for which the technique will be exact. However, their sum

$$1 + e^{jk\theta} = 2 \cos \frac{k\theta}{2} e^{jk\theta/2} \quad (24)$$

is not of the form  $(a + b\theta) e^{jk(\alpha + \beta\theta)}$ , and the sum of the (numerical) integrals is not equal to the (numerical) integral of the sum. However, if  $\mathbf{F}$  and  $\gamma$  are well behaved, the technique does in fact converge to the integral operator (and therefore becomes linear) in the limit of zero step size (Ref. 5).

Convergence tests for the case of scattering from a hyperboloid show that with this technique, incremental areas 2/3 of a square wavelength in size result in errors

more than 40 dB below the pattern maxima (Ref. 22). Similar tests of a program that uses Simpson's rule to evaluate a one-dimensional integral show that the areas must be at most 0.04 square wavelengths for the two-dimensional case, and possibly smaller (Ref. 22). Therefore, the number of integration points may be reduced by at least a factor of 16 relative to Simpson's rule. The plane reflector data presented in a previous section were computed with the step sizes given in Ref. 22. These data are an absolute test of numerical accuracy and are further confirmation of the convergence tests.

Although the complicated form of Eq. (23) seems to indicate that this technique would require an order of magnitude more time per data point than Simpson's rule, the fact is that in this type of problem the evaluation of  $e^{jx} = \cos x + j \sin x$  tends to dominate machine time. For example, an IBM 7094 will multiply or divide two numbers in about 10  $\mu$ s, or add two numbers in about 15  $\mu$ s (Ref. 25). However, computation of the sine and cosine to obtain the real and imaginary parts of  $e^{jx}$  requires 571  $\mu$ s with the library subroutine, or 281  $\mu$ s with the fast subroutine described in Appendix B (see Table B-1).

It is estimated that time per data point for this technique is increased by a factor of 2 to 4, relative to Simpson's rule, which means that the net reduction in total running time is a factor of 4 to 8. In practice, this technique can result in a reduction from 8 h to 1 or 2 h of computer time, so it is of substantial practical importance. For example, Slobin (Ref. 16) has reported that a program using Simpson's rule integration requires 1.3 min of IBM 360/75 computer time per output point to compute the scattered pattern of a 56-wavelength-diam tilted subreflector. Thus, 3.90 h would be required for 180 output points. These data have been obtained in 1.01 h of IBM 7094 Mod I computer time with the program described here. (The 1.01 h includes 9.8 min to evaluate the spherical-wave expansion of the incident fields at each of the integration grid points.) The 360/75 is faster than the 7094 Mod I by a factor of between 3 and 5; therefore the program described here is 11 to 19 times faster than this particular program that uses Simpson's rule. (It is interesting to note that the new technique works better on an old computer than the old technique on a new computer.)

It should be noted that in the case of rotational symmetry, when one integration may be performed analytically, these same data can be obtained by evaluating a one-dimensional integral (with a Bessel function in the

integrand) in about 6 min (Ref. 22). Therefore, it is grossly inefficient to evaluate a double integral in symmetrical cases, and the program described here should only be applied to problems involving asymmetrical configurations.

These convergence data are representative of the "low-gain" case in which the contributions from currents on the surface do not add in phase at any output point. A special case of considerable practical interest is the computation of the main lobe and first few sidelobes of the scattered pattern of a high-gain antenna such as a paraboloidal reflector. In this case, contributions are adding nearly in phase, i.e., the phase of the integrand is *very* slowly varying. Tests recently completed indicate that, in this case, integration points may be spaced at intervals of 20 wavelengths or more without seriously affecting the accuracy of the results. (Actually, in many practical cases closer spacing will be required to represent sufficient detail of the surface and incident-field data.) The results of these recent tests are similar to results reported by Allen for Filon's method and Gaussian quadrature for this case (Ref. 20), so the integration technique used here is nearly equivalent to the techniques in this situation. As mentioned previously, any of these methods may be improved by a Romberg type of modification in this case. The large spacings allowable between grid points means that the patterns of reflectors over 1000 wavelengths in diameter may be computed. However, beyond the first few sidelobes the situation begins to approach the "low-gain" case described earlier.

## VI. Comparison With Experimental Data and Other Results

An improved theory is important if it confirms the results of an approximate theory, since the approximation may then be used with confidence; it is also important if an improved theory corrects the results of an approximate theory. The application of spherical-wave theory developed in this report is considered valuable on both counts. A case in which the far-field approximation yields an incorrect result has already been presented; a case in which this approximation is adequate, except for the most precise calculations, is discussed in this section of the report.

A basic criterion in selecting this sample case was that it represent a real problem. The standard NASA/JPL Deep Space Network antenna is an 85-ft-diam paraboloid, with a Cassegrainian feed system, operated at

S-band. This configuration is also common for communications satellite terminals, so this sample case is clearly one that is of practical interest.

The object of the analysis was to compute the radiation pattern of the overall antenna, with particular emphasis on the pattern maximum that determines the antenna gain. The antenna is illustrated in Fig. 12. The design of this particular antenna is well documented (Ref. 26), and JPL provided experimental patterns of the fields scattered by the subreflector used with this configuration.

There is reason to question the application of the far-field approximation to the analysis of this antenna, because the subreflector is at  $1.65 D^2/\lambda$  of the primary feed, and the main reflector is at  $0.148 D^2/\lambda$  of the subreflector. Because the results of a far-field analysis generally agreed with experimental data, it was certain that the approximation was reasonably good, but the amount of error introduced was uncertain until the completion of the analysis presented in this report.

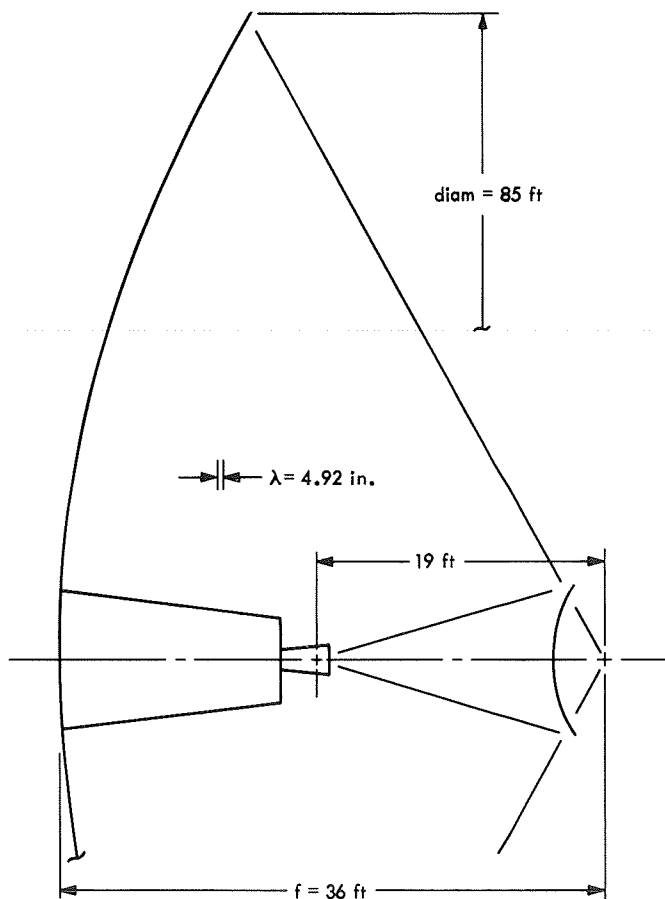


Fig. 12. Configuration of the 85-ft-diam antenna



The primary feed for this antenna was the same dual mode conical horn (Ref. 27) used for the plane reflector cases. The spherical-wave coefficients for this feed were obtained from an experimental pattern using the technique discussed in Appendix A; the incident-field patterns shown in Figs. 7 and 8 represent an evaluation of the spherical-wave expansion at  $\rho = \infty$ . Two parallel analyses were performed on the antenna. One analysis was made with the spherical-wave expansion FIELDS subroutine and the other with the far-field approximation FIELDS subroutine. The analyses were identical in all other respects.

The two computed scattered patterns of the subreflector (which consists of a vertex plate, a hyperboloidal section, and a conical flange) were virtually identical. This is not surprising, because the subreflector is near the  $2D^2/\lambda$  distance from the primary feed.

The experimental subreflector pattern was measured at a range that was nearly equal to the distance to the main reflector. In Fig. 13, these experimental data show good agreement with the pattern computed with the far-field approximation. The pattern computed using a spherical-wave expansion was in turn expanded in a series of spherical waves. In Fig. 14, the spherical-wave expansion, evaluated at the same radius at which the experimental data were measured, is compared to the experimental pattern. The agreement is even better than that in Fig. 13. The computed data closely follows the experimental pattern, demonstrating that the agreement between theory and experiment is outstanding.

The next step in the analysis was to use the subreflector pattern to illuminate the main paraboloidal reflector, again by the use of a spherical-wave expansion and the far-field approximation in the respective cases. The two computed main reflector patterns are compared in Fig. 15. Although the most noticeable difference is in the side lobes, this difference will generally be of little importance. Of greatest significance is that the peak of the main beam (i.e., the computed gain) differs by 0.127 dB in the two cases.

With the scale used in Fig. 15, a 0.127-dB difference is difficult to detect, and in many applications of computed antenna data it will be of no importance whatsoever. Therefore, for these applications (and for antennas of this approximate design) it has been shown that the faster and simpler far-field analysis is inadequate. For

example, the far-field analysis will be used for the feed displacement calculations to be presented shortly. Use of the far-field analysis is of particular importance in the case of development work involving a large amount of computer time.

However, there are cases in which a 0.127-dB difference is significant. Computed data have reached a level of sophistication where they are frequently weighted about equally with experimental data in calibrating large ground antennas (Ref. 28). An 85-ft antenna costs about one million dollars (Ref. 29), and it can be argued (Ref. 30) that an uncertainty of 0.127 dB in the gain of such an antenna is worth \$40,000. In fact, the costs of the time and effort spent to calibrate an antenna with this precision are even greater than \$40,000. Since the IBM 7094 computer time required for the two cases shown in Fig. 15 (including the spherical-wave expansion of the subreflector pattern) was 23.24 and 8.60 min for the spherical-wave and far-field analysis, respectively (a difference of about \$61.00 at the current price of computer time) it is probably safe to say that most engineers would choose the more accurate analysis. (Since the evaluation of the spherical-wave expansion on the integration grid is independent of the number of output points, the relative times would be more nearly equal for a larger number of output points.)

In all of the cases evaluated so far, rotational symmetry has been assumed. As a last example, the problem of an offset feed system will be considered to illustrate some of the capabilities of the program for asymmetric geometries.

The case considered is the offset feed geometry shown in Fig. 16. To provide a basis of comparison, the displacements along the dashed curve shown in the figure are the same as a case previously analyzed by Ruze (Ref. 31). However, the true subreflector pattern has been retained, thereby eliminating the need to assume a simple form for the illumination of the main reflector. In addition to the displacement translations, the feed system was rotated so that the edge angle of the main reflector remained nearly constant, to minimize spillover. The offset angle  $\beta$  is measured in the conventional manner. The computed patterns for a zero offset and two nonzero offsets are shown in Fig. 17. The analysis given by Ruze, which is based on scalar theory, includes the case of an illumination function of the form  $f(x) = 0.3 + 0.7(1 - x^2)$ , that approximates the reflector illumination in the case considered here.

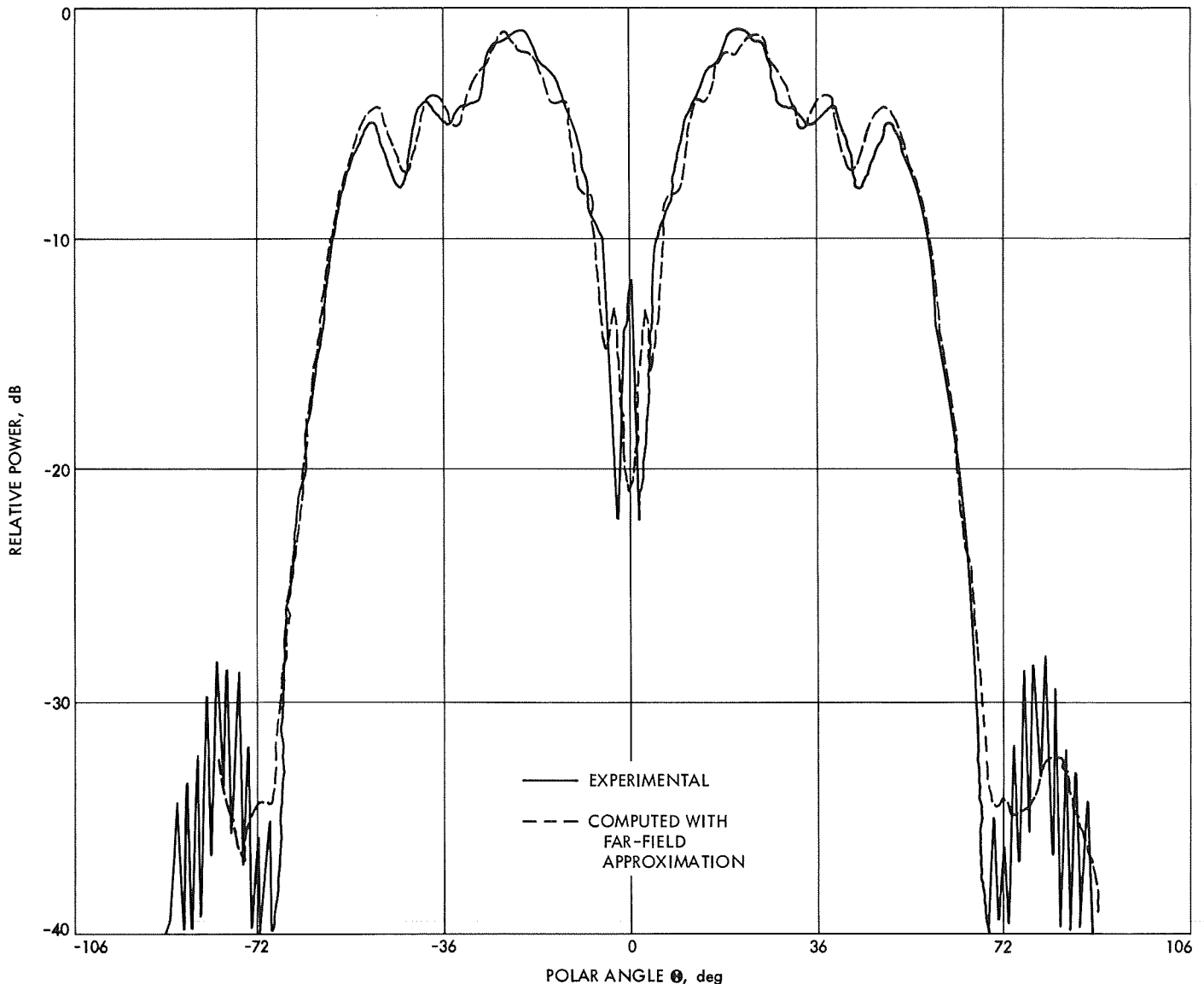
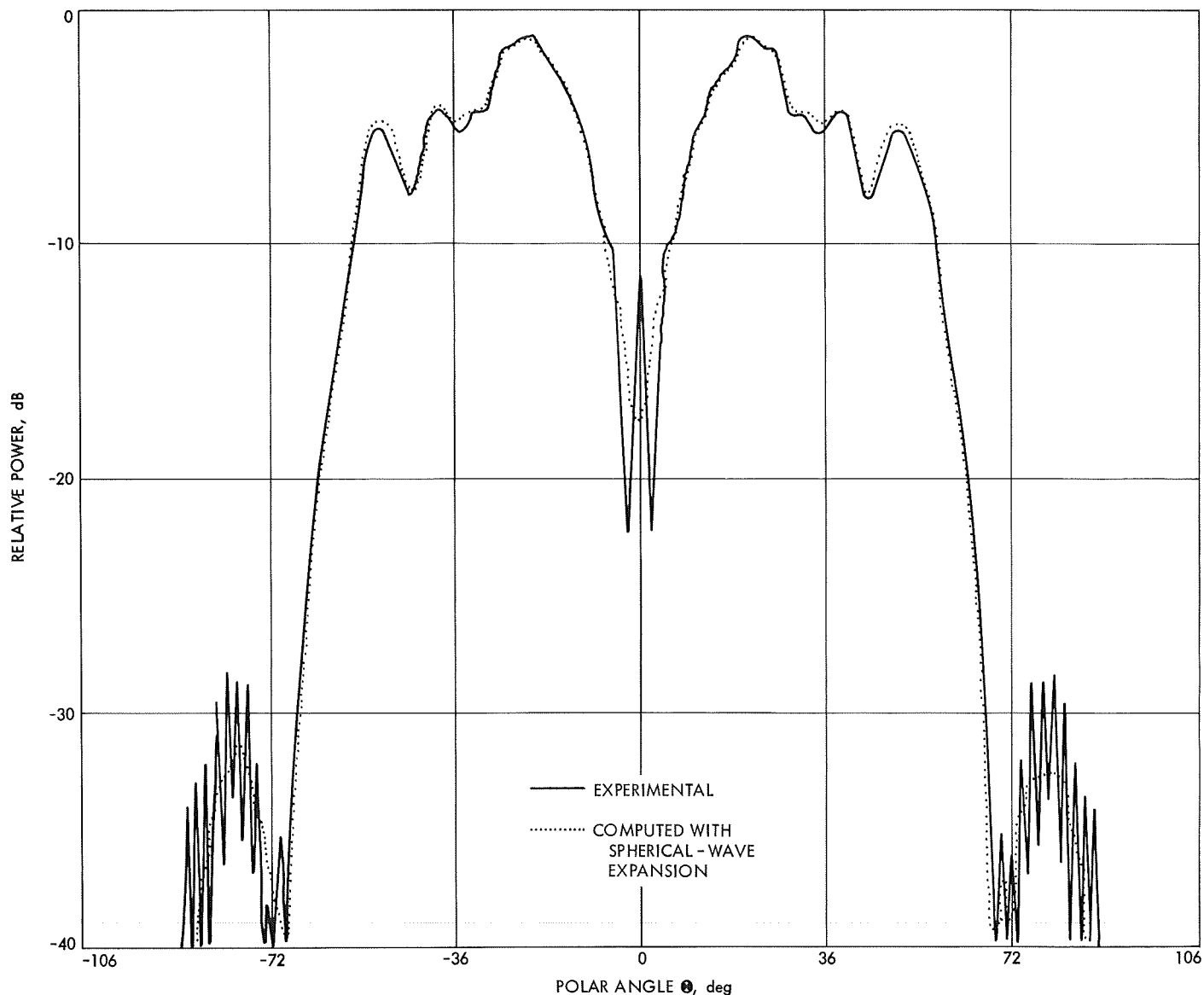


Fig. 13. Comparison of experimental and computed subreflector patterns; far-field approximation case

For this geometry and illumination, Ruze predicted a beam deviation factor of 0.84. The beam deviation factors of the results given in Fig. 17 are 0.835 and 0.840, for the 1- and 2-deg cases, respectively. The coma lobes agree within 1 dB with the values predicted by Ruze and the half and tenth-power beamwidths agree within 0.04 deg. The gain loss agrees (as well as it is possible to read the graphs given by Ruze) within approximately 0.1 dB.

This agreement is so complete that it might seem irrelevant to bother with the more rigorous analysis.

However, in addition to the confidence factor mentioned earlier, the approximate analysis has several restrictions: (1) it assumes a simple analytic illumination function, (2) the displacements *must* be along the specific curve shown in Fig. 16, and (3) the displacements are assumed to be small compared to the focal length. In the case considered in this section, these restrictions were met by design to obtain a comparison with independent data, but the program is capable of a case with (1) a very complicated illumination pattern (for example, a pattern with a central null produced by a vertex plate, including the phase pattern perturbations typical in this situation), (2) the displacements in any direction and, (3) a tilt



**Fig. 14. Comparison of experimental and computed subreflector patterns; spherical-wave expansion case**

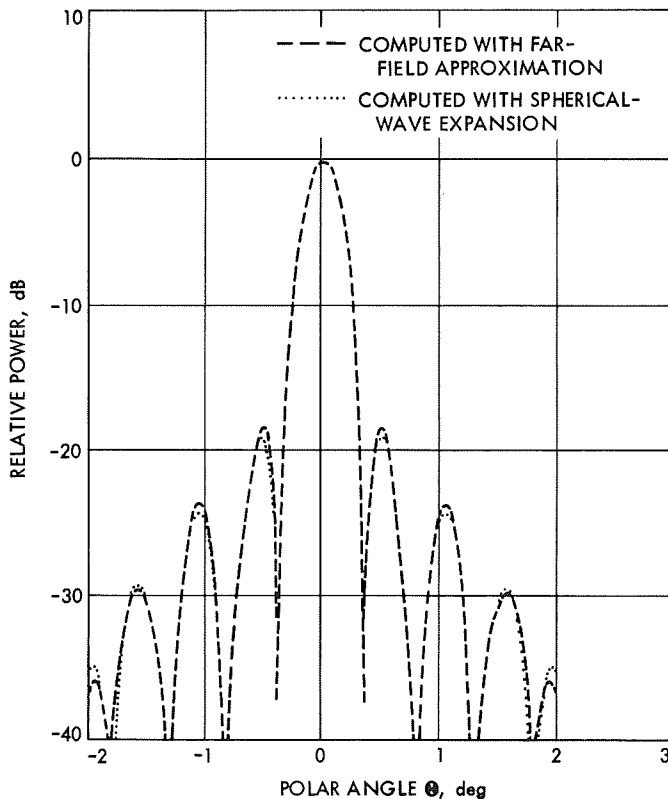
angle that is restricted to approximately 5 deg in the present program (the wavelength determines the allowable peak error that in turn determines the tilt angle as shown in Fig. 2), but that could be made arbitrarily large if more Fourier components were included, or if a special surface subroutine were written for the particular case of a tilted paraboloid.

## VII. Summary

The results shown in the previous section clearly demonstrate the usefulness of the physical-optics technique. The agreement between theoretical and experi-

mental data is excellent, and in the case of the 85-ft antenna, the application of a technique such as the integral-equation method to obtain the surface currents is virtually hopeless because of the large size of the reflector.

It was demonstrated in Section IV that the far-field approximation for the incident fields can lead to poor results, and that the spherical-wave representation corrects the results. The far-field expansion is still useful in many practical cases, but if precise results are required, and if the reflectors are in or close to the near-field region, the spherical-wave representation should be used.



**Fig. 15. Computed patterns of 85-ft-diam main reflector; comparison of far-field and spherical-wave expansion cases**

A Fourier type of representation of the scattering surface is efficient in the case of a slightly tilted figure of revolution, and in many cases three components are sufficient to accurately specify the surface.

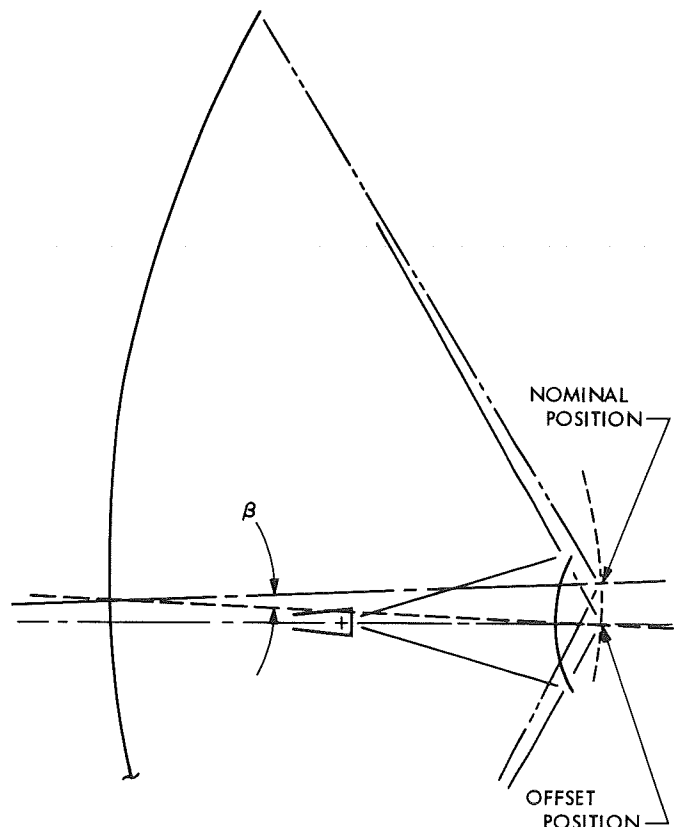
In the case of a low-gain asymmetrical reflector the program developed here is an order of magnitude faster than an earlier program; in the high-gain case the integration technique loses much of its advantage over prior methods, but the other timesaving techniques used still result in efficient operation. For symmetrical geometries, one integration should be performed analytically.

### VIII. Possible Improvements and Future Applications of Computer Program

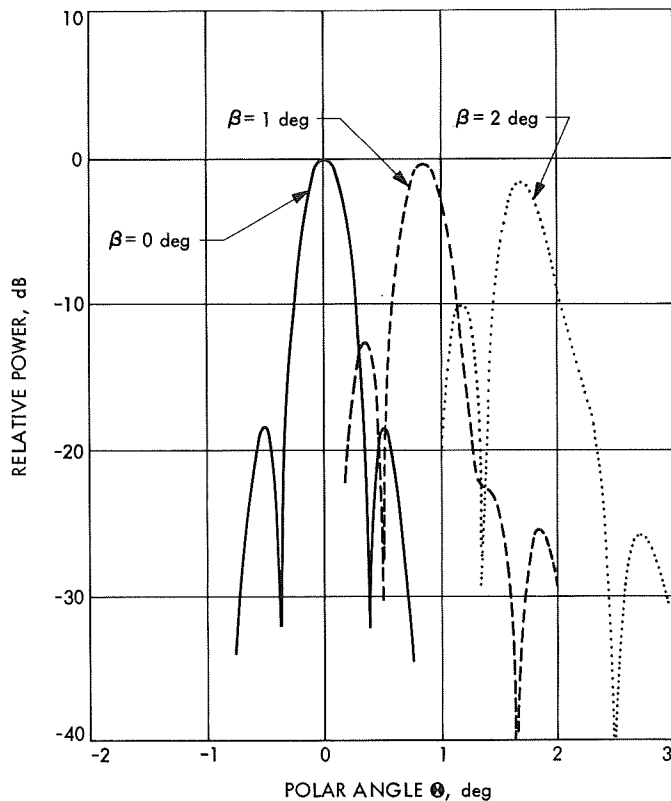
The computer program could be improved in several ways. For example, a Romberg type of procedure could be incorporated into the integration method. Also, in the existing program, when the edge of the reflector is not

coincident with a coordinate boundary, an approximate boundary must be constructed out of trapezoidal-like segments. An interesting extension of the integration method developed here would be to compute the scattered pattern of a triangular-shaped segment to be used in constructing edges that are at least continuous. Another possible improvement is to modify the program to accept an arbitrary number of Fourier components in the surface representation that would eliminate the small restriction on the reflector tilt angles.

The main area of future work lies in applying the scattering program. Parameter studies of near-field Cassegrainian antennas using actual experimental feed patterns is an obvious example. In all of the cases considered above, the spherical-wave coefficients of an incident-field pattern were obtained with the program described in Appendix D, but the scattering program will accept coefficients regardless of how they are obtained. For example, if the currents excited on a feed device, such as log-periodic wire antenna, are known (perhaps from the integral equation method), the spherical-wave coefficients may be obtained directly from the currents.



**Fig. 16. Geometry of offset-feed cases**



**Fig. 17. Computed patterns of 85-ft-diam main reflector with offset feed**

Also spherical-wave coefficients have been obtained analytically for cases such as a plane wave and an arbitrarily located and oriented dipole (Ref. 32). This dipole case is precisely the expansion required to compute the fields in the focal region of a reflector by the use of the same reciprocity technique recently used by Rusch (Ref. 21). This is a very interesting area for future work, since focal-region fields studied have involved assumptions and restrictions, whereas this method is, in principle, exact. Also, this method is applicable to nonparaboloidal antennas, such as shaped Cassegrainian antennas (Ref. 33), that have never been analyzed to obtain the focal-region fields. The reciprocal part of this problem is the computation of the pattern of reflectors with defocused feeds, and this is also a very interesting area for application of the scattering program, as discussed in the previous section.

The spherical-wave expansion program also has future applications independent of the scattering program. For example, a study is currently being pursued to apply spherical-wave expansions to the determination of correction factors for near-field gain measurements (Ref. 34). The usefulness of mode expansions in the solution of boundary-value problems has not been exhausted, and cases may be found in which the method is superior to the currently popular integral-equation method.

## Appendix A

### Spherical-Wave Expansions of Electromagnetic Fields

In a source-free, homogeneous, isotropic medium, spherical-wave solutions of Maxwell's equations are given by Stratton (Ref. 35):

$$\begin{aligned} \mathbf{m}_{\phi mn}^e = & \mp z_n(k\rho) \frac{m P_n^m(\cos \theta)}{\sin \theta} \frac{\sin}{\cos} m\phi \mathbf{i}_\theta \\ & - z_n(k\rho) \frac{\partial}{\partial \theta} P_n^m(\cos \theta) \frac{\cos}{\sin} m\phi \mathbf{i}_\phi \end{aligned} \quad (\text{A-1a})$$

$$\begin{aligned} \mathbf{n}_{\phi mn}^e = & n(n+1) \frac{z_n(k\rho)}{k\rho} P_n^m(\cos \theta) \frac{\sin}{\cos} m\phi \mathbf{i}_\rho \\ & + \frac{1}{k\rho} \frac{\partial}{\partial \rho} [\rho z_n(k\rho)] \frac{\partial}{\partial \theta} P_n^m(\cos \theta) \frac{\sin}{\cos} m\phi \mathbf{i}_\theta \\ & \pm \frac{1}{k\rho} \frac{\partial}{\partial \rho} [\rho z_n(k\rho)] \frac{m P_n^m(\cos \theta)}{\sin \theta} \frac{\cos}{\sin} m\phi \hat{\mathbf{i}}_\phi \end{aligned} \quad (\text{A-1b})$$

where

$e^{j\omega t}$  = time dependence (implicitly assumed)

$(\rho, \theta, \phi)$  = spherical coordinates (see Fig. 1)

$k = \omega(\epsilon\mu)^{1/2} = 2\pi/\lambda$  = propagation constant

$z_n(k\rho)$  = any solution of the spherical Bessel equation

$P_n^m(\cos \theta)$  = associated Legendre function

In the following derivation, an electromagnetic field is understood to be a pair of vector valued functions  $\mathbf{E}$  and  $\mathbf{H}$  (defined everywhere in a source-free, linear, homogeneous, isotropic region  $V$ ) that satisfies

$$\nabla \times \mathbf{E} \text{ and } \nabla \times \mathbf{H} \text{ exist everywhere in } V$$

and

$$\left. \begin{aligned} \nabla \times \mathbf{E} &= -j\omega\mu\mathbf{H} \\ \nabla \times \mathbf{H} &= j\omega\epsilon\mathbf{E} \end{aligned} \right\} \quad (\text{A-2a})$$

where  $\epsilon$  and  $\mu$  are physical constants, and  $\omega$  is a nonzero positive constant, and where

$$\left. \begin{aligned} \int_V \epsilon |\mathbf{E}|^2 dv \\ \int_V \mu |\mathbf{H}|^2 dv \end{aligned} \right\} \quad (\text{A-2b})$$

are defined and finite.

Since  $\nabla \cdot \nabla \times \mathbf{A} = 0$  for any vector  $\mathbf{A}$ , Eq. (A-2a) is equivalent to the assumption that Maxwell's equations are satisfied; by Poynting's theorem, Eq. (A-2b) is equivalent to the requirement that sources radiate finite power (assuming finite energy in any resonant fields).

The parallels between electromagnetic theory and the theory of complex variables are intriguing, and it is interesting to note that by Eq. (A-2a), the existence of the first derivative ( $\nabla \times \mathbf{E}$  and  $\nabla \times \mathbf{H}$ ) assures the existence of derivatives of all orders ( $\nabla \times \nabla \times \cdots \times \nabla \times \mathbf{E}$ , etc.). Other striking analogies are between the field integrals (which express the fields everywhere in a volume in terms of the values on the surface enclosing the volume) and Cauchy's theorem, and between spherical-wave expansions and Laurent series.

The objective of this section is as follows: given an electromagnetic field in a region  $V$  that consists of all space outside of a sphere of radius  $\rho_0$  (i.e., all sources are enclosed in this sphere), determine coefficients  $a_{\phi mn}^e$  and  $b_{\phi mn}^e$  so that everywhere in  $V$

$$\mathbf{E}(\rho, \theta, \phi) = - \sum_m \sum_n a_{\phi mn}^e \mathbf{m}_{\phi mn}^e + b_{\phi mn}^e \mathbf{n}_{\phi mn}^e \quad (\text{A-3a})$$

$$\mathbf{H}(\rho, \theta, \phi) = \frac{k}{j\omega\mu} \sum_m \sum_n a_{\phi mn}^e \mathbf{n}_{\phi mn}^e + b_{\phi mn}^e \mathbf{m}_{\phi mn}^e \quad (\text{A-3b})$$

Jones (Ref. 32) has shown that any electromagnetic field can be written in this form. Furthermore, since sources were assumed to be of a finite extent, only the solutions involving  $h_n^{(2)}(k\rho)$  (the spherical Hankel function of the second kind that satisfies the radiation condition and corresponds to an outward traveling wave, as discussed in Refs. 36 and 37) need be included.

Bouwkamp and Casimir (Ref. 38) obtained expressions for the coefficients in terms of the source currents; Kennaugh (Ref. 36) gives expressions in terms of the tangential  $\mathbf{E}$  and  $\mathbf{H}$  fields on a surface enclosing the sources; Jones (Ref. 32) includes expressions in terms of all components of either  $\mathbf{E}$  or  $\mathbf{H}$ . Since the data involved here are the tangential components of  $\mathbf{E}$  on a sphere of radius  $\rho_1 > \rho_0$ , a derivation for this case will be given. (Note that the case Jones considers is equivalent for the determination of TE wave coefficients but not for the determination of TM wave coefficients.)

As a first step, let  $a_{\sigma mn}^e$  and  $b_{\sigma mn}^e$  be any set of coefficients so that Eq. (A-3) is satisfied. (As mentioned above, Jones has shown at least one such set must exist.)

Define

$$a_{\sigma mn}^e \equiv a_{\sigma mn}^e h_n^{(2)}(k\rho_1) \quad (\text{A-4a})$$

$$b_{\sigma mn}^e \equiv b_{\sigma mn}^e \frac{1}{k\rho_1} \frac{\partial}{\partial \rho} [\rho h_n^{(2)}(k\rho)]_{\rho=\rho_1} \quad (\text{A-4b})$$

Then, since Eq. (A-3) is true everywhere in  $V$ , in particular, there must be equality of the tangential components at  $\rho = \rho_1$ :

$$E_\theta(\theta, \phi) = \sum_m \sum_n \mp a_{\sigma mn}^e \frac{mP_n^m(\cos \theta)}{\sin \theta} \frac{\sin}{\cos} m\phi + b_{\sigma mn}^e \frac{\partial}{\partial \theta} P_n^m(\cos \theta) \frac{\sin}{\cos} m\phi \quad (\text{A-5a})$$

$$E_\phi(\theta, \phi) = \sum_m \sum_n -a_{\sigma mn}^e \frac{\partial}{\partial \theta} P_n^m(\cos \theta) \frac{\cos}{\sin} m\phi \pm b_{\sigma mn}^e \frac{mP_n^m(\cos \theta)}{\sin \theta} \frac{\cos}{\sin} m\phi \quad (\text{A-5b})$$

where

$$E_\theta(\theta, \phi) \equiv \mathbf{E}(\rho_1, \theta, \phi) \cdot \mathbf{i}_\theta$$

$$E_\phi(\theta, \phi) \equiv \mathbf{E}(\rho_1, \theta, \phi) \cdot \mathbf{i}_\phi$$

The azimuthal components are separated out by the use of an ordinary Fourier expansion. (These equations are true for  $m > 0$ . For  $m = 0$ , an additional factor of 2 should be included but has been omitted for clarity.)

$$A_{m\sigma}^e(\theta) \equiv \int_0^{2\pi} E_\theta(\theta, \phi) \frac{\sin}{\cos} m\phi d\phi = -\pi \sum_n \mp a_{\sigma mn}^e \frac{mP_n^m(\cos \theta)}{\sin \theta} + b_{\sigma mn}^e \frac{\partial}{\partial \theta} P_n^m(\cos \theta) \quad (\text{A-6a})$$

$$B_{m\sigma}^e(\theta) \equiv \int_0^{2\pi} E_\phi(\theta, \phi) \frac{\cos}{\sin} m\phi d\phi = -\pi \sum_n -a_{\sigma mn}^e \frac{\partial}{\partial \theta} P_n^m(\cos \theta) \pm b_{\sigma mn}^e \frac{mP_n^m(\cos \theta)}{\sin \theta} \quad (\text{A-6b})$$

For the next step, the following relationship is needed:

$$\begin{aligned} \int_0^\pi \left[ \frac{mP_\ell^m(\cos \theta)}{\sin \theta} \pm \frac{\partial}{\partial \theta} P_\ell^m(\cos \theta) \right] \left[ \frac{mP_n^m(\cos \theta)}{\sin \theta} \pm \frac{\partial}{\partial \theta} P_n^m(\cos \theta) \right] \sin \theta d\theta = \\ \int_0^\pi \left[ \frac{m^2 P_\ell^m P_n^m}{\sin^2 \theta} + \frac{\partial}{\partial \theta} P_\ell^m \frac{\partial}{\partial \theta} P_n^m \right] \sin \theta d\theta \pm \int_0^\pi \left[ \frac{\partial}{\partial \theta} P_\ell^m \frac{mP_n^m}{\sin \theta} + \frac{mP_\ell^m}{\sin \theta} \frac{\partial}{\partial \theta} P_n^m \right] \sin \theta d\theta = \\ \begin{cases} 0 & \text{for } \ell \neq n \\ \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} n(n+1) & \text{for } \ell = n \end{cases} \quad (\text{A-7}) \end{aligned}$$

The equation above may be obtained by the performance of one integration by parts

$$\begin{aligned} \int_0^\pi \left[ \frac{\partial}{\partial \theta} P_\ell^m \frac{mP_n^m}{\sin \theta} + \frac{mP_\ell^m}{\sin \theta} \frac{\partial}{\partial \theta} P_n^m \right] \sin \theta d\theta = \\ \int_0^\pi \left[ \frac{\partial}{\partial \theta} P_\ell^m \frac{mP_n^m}{\sin \theta} \right] \sin \theta d\theta + [mP_\ell^m P_n^m]_0^\pi - \int_0^\pi \left[ \frac{\partial}{\partial \theta} P_\ell^m \frac{mP_n^m}{\sin \theta} \right] \sin \theta d\theta = 0 \quad (\text{A-8}) \end{aligned}$$

and application of a result given by Stratton (Ref. 35) (Section 7.13, Chapter VII).

Then it is easily verified that

$$\int_0^\pi \left[ \pm \frac{m P_l^m(\cos \theta)}{\sin \theta} A_{m_o}^e(\theta) + \frac{\partial}{\partial \theta} P_l^m(\cos \theta) B_{m_o}^e(\theta) \right] \sin \theta d\theta = \pi a_{o m l}' \frac{2}{2l+1} \frac{(\ell+m)!}{(\ell-m)!} \ell(\ell+1) \quad (\text{A-9a})$$

$$\int_0^\pi \left[ -\frac{\partial}{\partial \theta} P_l^m(\cos \theta) A_{m_o}^e(\theta) \mp \frac{m P_l^m(\cos \theta)}{\sin \theta} B_{m_o}^e(\theta) \right] \sin \theta d\theta = \pi b_{o m l}' \frac{2}{2l+1} \frac{(\ell+m)!}{(\ell-m)!} \ell(\ell+1) \quad (\text{A-9b})$$

Equations (A-4) through (A-9) may be rewritten in the expected form:

$$a_{o m l}' = \frac{1}{[z_n(k\rho_1)]^2} \frac{2n+1}{\pi 2n(n+1)} \frac{(n-m)!}{(\ell+m)!} \times \int_0^{2\pi} \int_0^\pi -\mathbf{m}_{o m n}^e \cdot \mathbf{E}_i(\rho_1, \theta, \phi)_{\text{tangential}} \sin \theta d\theta d\phi \quad (\text{A-10a})$$

$$b_{o m l}' = \frac{1}{\left\{ \frac{1}{k\rho_1} \frac{\partial}{\partial \rho} [\rho_1 z_n(k\rho_1)] \right\}^2} \frac{2n+1}{\pi 2n(n+1)} \frac{(n-m)!}{(n+m)!} \times \int_0^{2\pi} \int_0^\pi -\mathbf{n}_{o m n}^e \cdot \mathbf{E}_i(\rho_1, \theta, \phi)_{\text{tangential}} \sin \theta d\theta d\phi \quad (\text{A-10b})$$

Equation (A-10) not only gives an explicit form for the desired coefficients, but, by the well known properties of trigonometric and Legendre function expansions, the coefficients are uniquely determined. Therefore, this illustrates the fact that an electromagnetic field as defined earlier is uniquely determined by the value of the tangential  $E$  field on any sphere of radius  $\rho_1 > \rho_0$ .

The spherical-wave expansion program (orthogonality version) given in Appendix D evaluates Eq. (A-10) to obtain  $a_{o m n}'$  and  $b_{o m n}'$ . That is, it is assumed that the azimuthal Fourier expansion of Eq. (A-6) has already been performed, and the input data to the spherical-wave expansion program are the (tabular) functions  $A_{m_o}^e(\theta)$  and  $B_{m_o}^e(\theta)$ . Because successive cases with different values of  $m$  may be evaluated, in principle an arbitrary pattern can be expanded. However, the  $m = 1$  case is of particular importance, as mentioned previously, which is why the program was written to handle only a single azimuthal component at one time. The odd components have also been neglected to avoid unnecessary complication, since the vast majority of problems may be analyzed using linear polarization. In the case of far-field input data,

Eq. (A-4) may be rewritten with the asymptotic form of the Hankel functions, thereby yielding for  $\rho_1 \rightarrow \infty$

$$a_{o m n}' = a_{o m n}^e j^{n+1} \frac{\exp(-jk\rho_1)}{k\rho_1} \quad (\text{A-11a})$$

$$b_{o m n}' = b_{o m n}^e j^n \frac{\exp(-jk\rho_1)}{k\rho_1} \quad (\text{A-11b})$$

The scattering program implicitly assumes Eq. (A-11) so it accepts the wave coefficients as output from the spherical-wave expansion program (it is assumed a far-field pattern is input to the spherical-wave expansion program).

Since the solution is known to exist and because it is unique, it is also possible to solve Eq. (A-6) directly. It is well known that if all sources are enclosed in the sphere of radius  $\rho_0$ , only modes for which  $n \leq k_0$  will make a significant contribution to the field (Refs. 39-41). Therefore, Eq. (A-6) can be solved as a system of linear equations to obtain the coefficients. Again, with only even terms and a single  $m$  component, the  $m$  and  $o$  subscripts can be



dropped and the matrix equation can be written:

$$\begin{bmatrix} A(\theta_1) \\ B(\theta_1) \\ A(\theta_2) \\ B(\theta_2) \\ \vdots \\ A(\theta_N) \\ B(\theta_N) \end{bmatrix} = \pi \begin{bmatrix} F_1^m(\theta_1) & -G_1^m(\theta_1) & F_2^m(\theta_1) & \cdots & -G_N^m(\theta_1) \\ G_1^m(\theta_1) & -F_1^m(\theta_1) & G_2^m(\theta_1) & \cdots & -F_N^m(\theta_1) \\ F_1^m(\theta_2) & -G_1^m(\theta_2) & F_2^m(\theta_2) & \cdots & -G_N^m(\theta_2) \\ G_1^m(\theta_2) & -F_1^m(\theta_2) & G_2^m(\theta_2) & \cdots & -F_N^m(\theta_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_1^m(\theta_N) & -G_1^m(\theta_N) & F_2^m(\theta_N) & \cdots & -G_N^m(\theta_N) \\ G_1^m(\theta_N) & -F_1^m(\theta_N) & G_2^m(\theta_N) & \cdots & -F_N^m(\theta_N) \end{bmatrix} \begin{bmatrix} a'_1 \\ b'_1 \\ a'_2 \\ b'_2 \\ \vdots \\ a'_N \\ b'_N \end{bmatrix} \quad (\text{A-12})$$

where

$$F_n^m(\theta) \equiv \frac{m P_n^m(\cos \theta)}{\sin \theta}$$

$$G_n^m(\theta) \equiv \frac{\partial}{\partial \theta} P_n^m(\cos \theta)$$

$$N \simeq k\rho_0$$

Since  $F_n^m(0 \text{ deg}) = G_n^m(0 \text{ deg})$  and  $F_n^m(180 \text{ deg}) = -G_n^m(180 \text{ deg})$ , the values  $\theta = 0$  and  $180 \text{ deg}$  would make the above matrix singular; therefore, they must not be used as data points. Even excluding these values, the first attempts to invert this system of equations indicated that the matrix was extremely ill-conditioned (on physical grounds it can not be singular if the  $\theta$  values are distinct and if  $\theta = 0$  and  $180 \text{ deg}$  are excluded). It was subsequently determined that the matrix was ill-conditioned because the data points were concentrated in the region between  $0$  and  $40 \text{ deg}$  (where  $A(\theta)$  and  $B(\theta)$  had significant amplitudes). When the data points were equally spaced between  $0$  and  $180 \text{ deg}$ , the matrix was easily inverted. The program that was used to accomplish this, the spherical-wave expansion program (linear equation version), is also presented in Appendix D. When properly used, both programs yielded the same results except for modes containing a negligible fraction of the total power.

The linear equation method is included here primarily because it may be easily generalized to be valid when the fields are known on any surface, whereas the formulation of the orthogonality technique is valid only over the surface of a sphere. Furthermore, the linear equation method provides an interesting numerical check on the other program. However, the orthogonality method is more convenient for a number of reasons: (1) it is faster to perform the numerical integration than to invert the matrix, (2) the number of mode coefficients is not restricted to the number of input  $\theta$  values, and (3) Parseval's formula can be used to show that a sufficient number of modes have been considered, rather than the well documented but essentially qualitative arguments leading to the  $N \simeq k\rho$  relationship.

The equations for obtaining wave coefficients derived here differ slightly from those obtained previously. The main difference is that these equations have been programmed and used to obtain wave expansions of three kinds of patterns—analytical, numerically computed, and experimental. Previous work has been restricted to interesting but idealized cases in which the coefficients were obtained analytically in closed form; e.g., Potter (Ref. 41) found the coefficients for a circularly symmetric optimum illumination pattern; Kennaugh and Ott (Ref. 42) found the wave excited by a plane wave normally incident on a paraboloid; Jones (Ref. 32) gives the coefficients for a plane wave and for an arbitrarily located electric dipole.

## Appendix B

### Fast Trigonometric Subroutine

The machine-language function subroutine\* EXPJX was written for the IBM 7094 computer and returns the complex-valued result  $e^{jx} = \cos x + j \sin x$ , for real  $x$ , in one-half the time required by the IBM library subroutine, and with only slight reduction in accuracy.

A table lookup technique is employed where 64 values of  $\sin x_m$  at  $x_m = m(\pi/128) + (\pi/256)$ ;  $m = 0, 1, \dots, 63$  are built into the subroutine.

The basic algorithm used is as follows:

- (1)  $\sin(-x) = -\sin(x)$ ,  $\cos(-x) = \cos(x)$ , assume  $x \geq 0$ .
- (2) Write  $x = n(\pi/128) + r(\pi/128)$  where  $n$  is an integer, and  $1 > r \geq 0$ .
- (3) Reduce  $n \bmod 256$  to discard multiples of  $2\pi$ .
- (4) Write  $n = 64q + m$ , where  $63 \geq m \geq 0$ . Then  $q + 1$  is the quadrant of the argument, and  $m$  is the index of the tabular value  $x_m$  closest to  $x$  (reduced to the 1st quadrant).
- (5) Define  $\Delta x = x - x_m = (r - 1/2)\pi/128$ ; and  $|\Delta x| < \pi/265$ .
- (6) Use  $\cos x_m = \sin x_{63-m}$ , the expansions

$$\sin(x_m + \Delta x) = \sin x_m \cos \Delta x + \cos x_m \sin \Delta x$$

$$\simeq \sin x_m \left(1 - \frac{\Delta x^2}{2!}\right) + \cos x_m \left(\Delta x - \frac{\Delta x^3}{3!}\right)$$

$$\cos(x_m + \Delta x) = \cos x_m \cos \Delta x - \sin x_m \sin \Delta x$$

$$\simeq \cos x_m \left(1 - \frac{\Delta x^2}{2!}\right) - \sin x_m \left(\Delta x - \frac{\Delta x^3}{3!}\right)$$

and the approximation

$$x - \frac{\Delta x^3}{3!} \simeq \Delta x(1 - 0.16 \times 10^{-4})$$

for  $|\Delta x| < (\pi/256)$  to compute  $\sin x$  and  $\cos x$  in the 1st quadrant.

- (7) Use the quadrant index  $q$  and the sign of the original argument to obtain  $\sin x$  and  $\cos x$  for the unreduced (true) argument, and return the complex number  $(\cos x, \sin x)$ .

The reason for choosing 64 (or in general  $2^n$ ) tabular values is that a single multiplication of the input argument  $x(128/\pi)$  results in a binary number in which the fractional part is  $r$ , the next six higher order bits are  $m$ , the next two bits are  $q$ , and higher order bits represent multiples of  $2\pi$ . Therefore, steps (2), (3), and (4) may be accomplished very quickly by one multiplication followed by shifting and masking operations.

The MAP subroutine, which was named EXPJX, requires a total of 167 (decimal) storage locations, compared to 123 locations for the IBM Fortran IV library routine FCSN.

An accuracy and timing test was made in which results from EXPJX and FCSN were compared to the IBM double precision Fortran IV library routine FDSC. The library routine was called twice to obtain values for both  $\sin x$  and  $\cos x$ . Values were computed for  $50 \times 10^3$  random arguments uniformly distributed over the range  $(-5 \text{ to } 5\pi)$ , with results as shown in Table B-1.

The relative accuracy of EXPJX becomes poor for small values of  $\sin x$  or  $\cos x$ , and for arguments  $|x| > 8\pi$  the loss of significance in the reduced argument will increase the errors.

The subroutine of Printout B-1 has been submitted for distribution through SHARE (SHARE library distribution number SDA 3534), and COSMIC (Program NPO 10439).

**Table B-1. Performance statistics for fast trigonometric subroutine**

Program	Absolute error (rms)	Maximum absolute error	Average speed, $\mu s$
EXPJX	$8.2 \times 10^{-8}$	$3.7 \times 10^{-7}$	281
FCSN	$0.28 \times 10^{-8}$	$0.13 \times 10^{-7}$	571 <sup>a</sup>
<sup>a</sup> For both sine and cosine.			

\*J. Hatfield of JPL Section 314, and H. Thacher of Notre Dame University were associated with the author in the development of this subroutine.

# Printout B-1. Computer printout of fast trigonometric subroutine

```

$IBMAP EXPJX LIST,REF,M94,()OK
      LBL EXPJX,BEGIN,
      TTL COMPLEX EXPONENTIAL SUBROUTINE...@EXPJX@
      REM *****
      REM *****
      REM ***** COMPLEX EXPONENTIAL SUBROUTINE...@EXPJX@*****
      REM *****
      REM *****
      SPACE 1
      ENTRY EXPJX
N      SET 64          SIZE OF SINE-COSINE TABLE
      SPACE 2
      REM THIS SUBROUTINE COMPUTES
      SPACE 1
      REM EXP(J*X)=COS(X)+J*SIN(X)
      SPACE 1
      REM GIVEN THE FLOATING PT. VALUE, X.
      REM THE COS(X) AND SIN(X) ARE TABULAR EVALUATED AND @EXPJX@
      REM RETURNS WITH THEIR RESPECTIVE RESULTS IN THE AC. AND MQ.
      SPACE 3
      REM PROCEDURE...
      REM 1. COMPUTE Y=(1/DTHETA)*X AND SAVE SIGN OF X
      REM 2. UNFLOAT Y WITH INTEGER IN AC. AND FRACTION IN MQ
      REM 3. UNPACK INTEGER SO THAT QUAD. NO. AND TABLE INDEX
      REM ARE BITS 28-29 AND 30-35 RESPECTIVELY
      REM 4. COMPUTE DX=ABS(FRACTION OF Y)*DTHETA-DTHETA/2
      REM 5. COMPUTE T1=SIN(DX) AND T2=COS(DX) USING SERIES
      REM EVALUATION (AT MOST TWO TERMS)
      REM 6. COMPUTE F1=SIN(X@)*T1+COS(X@)*T2
      REM F2=COS(X@)*T1-SIN(X@)*T2
      REM WHERE SIN(X@) AND COS(X@) ARE OBTAINED FROM
      REM IT@S CORRESPONDING TABLE X@=X-DX
      REM 7. FOR THE APPROPRIATE QUADRANT COMPUTE..
      REM 1ST QUAD. SIN(X)=(SIGN) F1
      REM COS(X)=F2
      REM 2ND QUAD. SIN(X)=(SIGN) F2
      REM COS(X)=-F1
      REM 3RD QUAD. SIN(X)=-(SIGN) F1
      REM COS(X)=-F2
      REM 4TH QUAD. SIN(X)=-(SIGN) F2
      REM COS(X)=F1
      REM 8. FLOAT SIN(X) AND COS(X)
      SPACE 3
EXPJX SXA EXIT1,4
      SXA EXIT1+1,1
      SXA EXIT1+2,2
      SPACE 1
      LDQ* 3,4          FETCH X (IN RADIANS)
      STQ SIGN
      FMP 10VDT          COMPUTE X/DTHETA
      SSP
      UFA =0233000000000 UNFLOAT IT (I IN ACC., F IN MQ.)
      RQL 8
      LRS 0              SCALE F AT BIT 0 IN MQ.
      ANA =0377          SET SIGN OF F POSITIVE
      LGR 6              MASK I FOR QUADRANT AND INDEX NUMBERS
      PAC 0,1            XR1=QUADRANT NUMBER
      PXA 0,0
      LGL 6
      PAC 0,2            XR2=INDEX NUMBER FOR SINE
      PAX 0,4            XR4=INDEX NUMBER FOR COSINE
      MPY DT             COMPUTE F@=F*(PI/2)/64
      SUB DTOV2
      STO DX             SAVE DELTAX=F@-(PI/2)/128
      ARS 1
      CHS
      XCA
      MPY DX
      ALS 1             CHANGE SCALE FROM BIT 2 TO BIT 1

```

# Printout B-1 (contd)

	ADD	=181	
	STO	T1	T1=1-DELTAX**2/2 (COS(DELTAX))
	LDQ	SINX,2	
	MPY	T1	
	STO	F1	F1=SIN(Xa)*T1
	LDQ	COSX,4	
	MPY	T2	
	ADD	F1	
	STO	F1	F1=SIN(Xa)*T1+COS(Xa)*T2
	LDQ	SINX,2	
	MPY	T2	
	STO	F2	F2=SIN(Xa)*T2
	LDQ	COSX,4	
	MPY	T1	
	SUB	F2	
	STO	F2	F2=COS(Xa)*T1-SIN(Xa)*T2
	TRA	*+1,1	
	TRA	QUAD1	
	TRA	QUAD2	
	TRA	QUAD3	
	TRA	QUAD4	
	SPACE	1	
QUAD1	CLA	F1	
	LDQ	SIGN	
	LLS	0	
	LDQ	F2	
	TRA	EXIT	(SIN(X)=SIGN.F1, COS(X)=F2)
	SPACE	1	
QUAD2	LDQ	F2	
	CLA	SIGN	
	LRS	0	
	CLS	F1	
	XCA		
	TRA	EXIT	(SIN(X)=SIGN.F2, COS(X)=-F1)
	SPACE	1	
QUAD3	LDQ	F1	
	CLA	SIGN	
	CHS		
	LRS	0	
	CLS	F2	
	XCA		
	TRA	EXIT	(SIN(X)=-SIGN.F1, COS(X)=-F2)
	SPACE	1	
QUAD4	CLA	F2	
	LDQ	SIGN	
	LLS	0	
	CHS		
	LDQ	F1	(SIN(X)=-SIGN.F2, COS(X)=F1)
	SPACE		
EXIT	STO	SIN(X)	
	CLA	=0202	PACK IN EXPONENT FOR BIT 2 SCALE
	LLS	0	BUT BE SURE AND RETAIN SIGN OF
	LRS	8	COS(X)... THE RESULT IS AN
	XCA		UNNORMALIZED FL. PT. NO. (IN ACC.)
	FAD	=0	NORMALIZE COS(X) AND SAVE
	STO	COS(X)	THE FLOATING POINT RESULT
	LDQ	SIN(X)	
	CLA	=0202	PACK IN EXPONENT FOR BIT 2 SCALE AND
	LLS	0	MAKE SURE THE SIGN OF SIN(X) IS
	LRS	8	RETAINED...THE RESULT IS AN
	XCA		UNNORMALIZED FL. PT. NO. (IN ACC.)
	FAD	=0	NORMALIZE SIN(X) AND
	XCA		PLACE RESULT IN MQ.
	CLA	COS(X)	PLACE COS(X) IN ACC.
EXIT1	AXT	** ,4	
	AXT	** ,1	
	AXT	** ,2	
	TRA	1,4	
	EJECT		
DT	DEC	0.0245432999B1	(PI/2)*(1/64)*0.999984 (SCALED AT BIT 1)

# Printout B-1 (contd)

DTOV2	DEC	0.0122716499B1	(PI/2)*(1/64)/2*0.999984(SCALED AT BIT 1)
10VDT	DEC	40.7436654	1/DTHETA (FLOAT. PT.)
DX	PZE	**	DELTA X
SIGN	PZE	**	SIGN OF X
T1	PZE	**	COSINE OF DELTAX (SCALED AT BIT 1)
T2	EQU	DX	SINE OF DELTAX (SCALED AT BIT 1)
F1	PZE	**	SIN(X)*T1+COS(X)*T2 (SCALED AT BIT 2)
F2	PZE	**	COS(X)*T1-SIN(X)*T2 (SCALED AT BIT 2)
SIN(X)	PZE	**	ABSOLUTE VALUE OF SIN(X)
COS(X)	PZE	**	ABSOLUTE VALUE OF COS(X)
EJECT			
REM			THE FOLLOWING TABLE CONTAINS FIXED-POINT VALUES
REM			SCALED AT BIT 1
SINX	NULL		
	DEC	0.0122715383B1	
	DEC	0.0368072229B1	
	DEC	0.0613207363B1	
	DEC	0.0857973123B1	
	DEC	0.1102222073B1	
	DEC	0.1345807085B1	
	DEC	0.1588581433B1	
	DEC	0.1830398880B1	
	DEC	0.2071113762B1	
	DEC	0.2310581083B1	
	DEC	0.2548656596B1	
	DEC	0.2785196894B1	
	DEC	0.3020059493B1	
	DEC	0.3253102922B1	
	DEC	0.3484186803B1	
	DEC	0.3713171940B1	
	DEC	0.3939920401B1	
	DEC	0.4164295601B1	
	DEC	0.4386162385B1	
	DEC	0.4605387110B1	
	DEC	0.4821837721B1	
	DEC	0.5035383837B1	
	DEC	0.5245896827B1	
	DEC	0.5453249884B1	
	DEC	0.5657318108B1	
	DEC	0.5857978575B1	
	DEC	0.6055110414B1	
	DEC	0.6248594881B1	
	DEC	0.6438315429B1	
	DEC	0.6624157776B1	
	DEC	0.6806009978B1	
	DEC	0.6983762494B1	
	DEC	0.7157308253B1	
	DEC	0.7326542717B1	
	DEC	0.7491363945B1	
	DEC	0.7651672656B1	
	DEC	0.7807372286B1	
	DEC	0.7958369046B1	
	DEC	0.8104571983B1	
	DEC	0.8245893028B1	
	DEC	0.8382247056B1	
	DEC	0.8513551931B1	
	DEC	0.8639728561B1	
	DEC	0.8760700942B1	
	DEC	0.8876396204B1	
	DEC	0.8986744657B1	
	DEC	0.9091679831B1	
	DEC	0.9191138517B1	
	DEC	0.9285060805B1	
	DEC	0.9373390119B1	
	DEC	0.9456073254B1	
	DEC	0.9533060404B1	
	DEC	0.9604305194B1	
	DEC	0.9669764710B1	
	DEC	0.9729399522B1	
	DEC	0.9783173707B1	

**Printout B-1 (contd)**

	DEC	0.9831054874B1
	DEC	0.9873014182B1
	DEC	0.9909026354B1
	DEC	0.9939069700B1
	DEC	0.9963126122B1
	DEC	0.9981181129B1
	DEC	0.9993223846B1
	DEC	0.9999247018B1
COSX	EQU	SINX+N-1
	END	

## Appendix C

### Tilted Reflector Program

The tilted reflector program was written for an IBM 1620 computer and outputs a deck of cards containing the quantities (see Eq. 12):

$$\begin{array}{ccc} a_0(\theta), & a_1(\theta), & a_2(\theta), \\ \frac{\partial a_0}{\partial \theta}, & \frac{\partial a_1}{\partial \theta}, & \frac{\partial a_2}{\partial \theta} \end{array}$$

The input data are tabular values  $\rho'(\theta')$  describing a surface of revolution, as illustrated in Fig. C-1. Also specified are the translations to the new origin XF and ZF; a coordinate rotation is specified by the intersection point ZROT.

The procedure used is as follows:

- (1) By the use of a straightforward coordinate transformation, the data

$$\begin{aligned} \rho'(\theta', 0 \text{ deg}) &= \rho'(\theta', 90 \text{ deg}) \\ &= \rho'(\theta', 180 \text{ deg}) \\ &= \rho'(\theta') \end{aligned}$$

are mapped into the new system for each input  $\theta'$  value. Note that the cuts  $\phi' = 0$  or  $180$  deg map into  $\phi = 0$  or  $180$  deg, but in general  $\phi' = 90$  deg maps into a cut which will be identified by  $\phi_{90}(\theta)$ . The result of this step is data

$$\rho(\theta, 0 \text{ deg}), \quad \rho(\theta, \phi_{90}), \quad \rho(\theta, 180 \text{ deg})$$

that are known at a set of  $\theta$  values that is in general different for each  $\phi$  cut, and different from the set of desired output values. The remaining steps (2) through (4) are performed for each desired output value  $\theta_0$ .

- (2) A cubic interpolation polynomial in  $\theta$  is fitted locally at four points to the functions  $\rho(\theta, 0 \text{ deg})$ ,  $\rho(\theta, \phi_{90})$ ,  $\rho(\theta, 180 \text{ deg})$  and  $\phi_{90}(\theta)$ . These polynomials and their derivatives are evaluated at  $\theta = \theta_0$  to yield  $\phi_{90}(\theta_0)$  and  $\partial \phi_{90} / \partial \theta$  at  $\theta = \theta_0$ ; and  $\rho(\theta_0, 0 \text{ deg})$ ,  $\rho(\theta_0, \phi_{90})$ ,  $\rho(\theta_0, 180 \text{ deg})$  and  $d\rho/d\theta$  at  $\phi = 0$ ,  $\phi_{90}$ , and  $180$  deg.

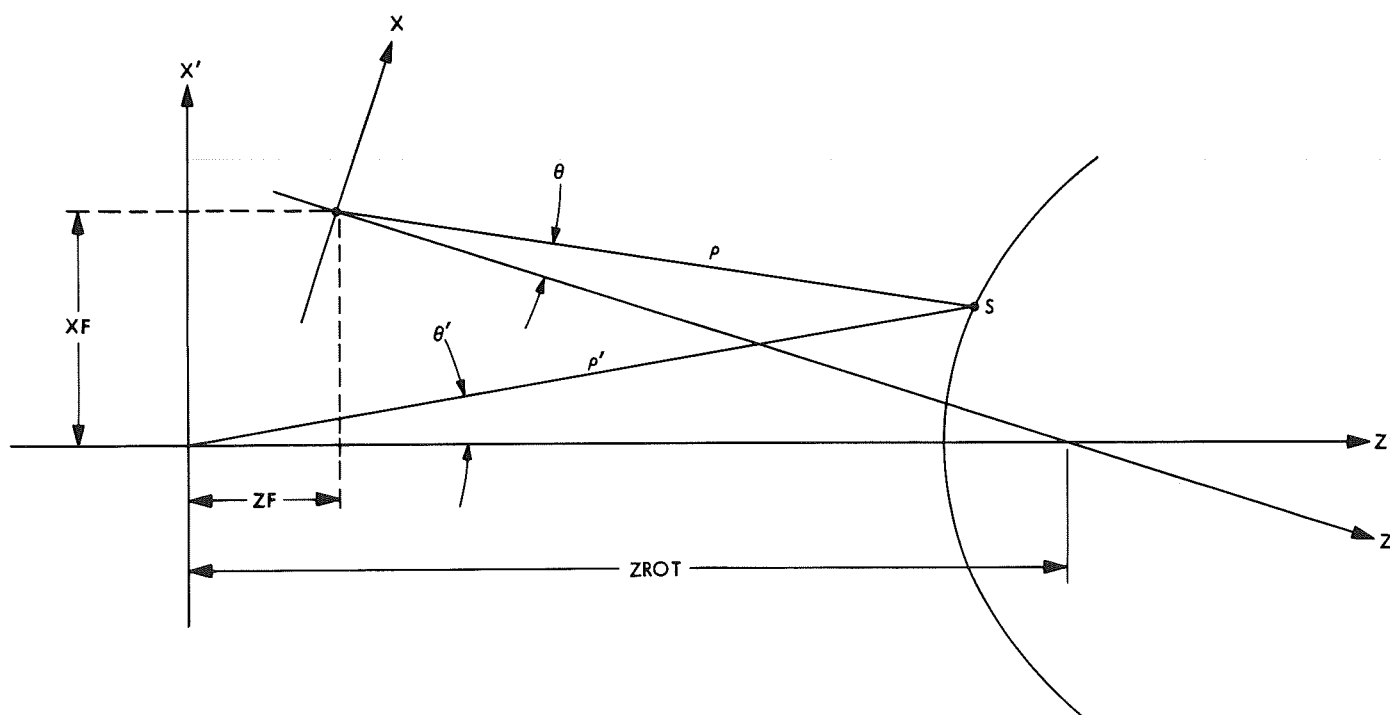


Fig. C-1. Reflector coordinate transformation

(3) Given the assumed form

$$\rho(\theta, \phi) = a_0(\theta) + a_1(\theta) \cos \phi + a_2(\theta) \cos 2\phi$$

it follows that

$$\begin{bmatrix} \rho(\theta_0, 0) \\ \rho(\theta_0, \phi_{90}) \\ \rho(\theta_0, 180) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \cos \phi_{90} & \cos 2\phi_{90} \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a_0(\theta_0) \\ a_1(\theta_0) \\ a_2(\theta_0) \end{bmatrix}$$

By the use of  $\phi_{90}(\theta_0)$  from step (2), the program then inverts the matrix and obtains  $a_0(\theta_0)$ ,  $a_1(\theta_0)$ , and  $a_2(\theta_0)$ .

For the matrix to be well-conditioned, it is sufficient that  $\phi_{90} \simeq 90$  deg. If  $\phi_{90}$  lies outside the range of 60 to 120 deg the program sets  $a_2(\theta) = 0$  and solves the two-dimensional matrix problem for  $a_0(\theta)$  and  $a_1(\theta)$ .

(4) Also, from the assumed form,

$$\frac{\partial \rho}{\partial \theta} = \frac{\partial a_0}{\partial \theta} + \frac{\partial a_1}{\partial \theta} \cos \phi + \frac{\partial a_2}{\partial \theta} \cos 2\phi$$

$$\frac{\partial \rho}{\partial \phi} = -a_1(\theta) \sin \phi - 2a_2(\theta) \sin 2\phi$$

By the use of the relationship

$$\begin{aligned} \frac{\partial \rho}{\partial \theta} &= \frac{d\rho}{d\theta} - \frac{\partial \rho}{\partial \phi} \frac{\partial \phi}{\partial \theta} \\ &= \frac{d\rho}{d\theta} + [a_1(\theta) \sin \phi + 2a_2(\theta) \sin 2\phi] \frac{\partial \phi}{\partial \theta} \end{aligned}$$

and the values of  $d\rho/d\theta$  and  $\partial\phi_{90}/\partial\theta$  at  $\theta = \theta_0$  obtained in step (2), values of  $\partial\rho/\partial\theta$  are obtained at  $\phi = 0$  deg,  $\phi_{90}$ , and 180 deg. Then the same matrix inverse obtained in step (3) is used to yield  $\partial a_0/\partial\theta$ ,  $\partial a_1/\partial\theta$  and  $\partial a_2/\partial\theta$ .

The input data for the tilted reflector program are shown in Table C-1; the parameters are defined below:

TITLE any alphanumeric statement

XF, ZF, ZROT see Fig. C-1

T, R  $\theta', \rho'$  (Note that  $\theta'$  must be monotonically decreasing with its index.)

9999.0 any number  $> 400$  (A last card indicator. NMAX must be  $\leq 200$ . If the input consists of more than 200 cards, the program will type out an error message; the program can be reset to read title card.)

NMAX number of  $\theta$  values desired

TOUT desired  $\theta$  values

The program prints out the input data ( $\rho', \theta'$ ) and transformed data ( $\rho, \theta$ ) at  $\phi = 0, 180$  deg, and  $\phi_{90}$ . Then  $\rho$  and  $\partial\rho/\partial\theta$  are printed out at the desired  $\theta$  values (Printout C-1).

The Fourier coefficients and their derivatives are punched but not printed.

Table C-1. Input data for the tilted reflector program

Card	Parameters	Format
1	TITLE	20A4
2	XF ZF ZROT	8F10.0
3	T(1) R(1)	20X, 2F10.0
.	T(2) R(2)	. .
.	. .	. .
.	. .	. .
.	. .	. .
NMAX + 2	T(NMAX) R(NMAX)	20X, 2F10.0
NMAX + 3	9999.0	20X, F10.0
NMAX + 4	MMAX	10I5
NMAX + 5	TOUT(1) TOUT(2) . . . TOUT(8)	8F10.0
.	TOUT(9) TOUT(10) . . .	.
.	.	.
.	. . . TOUT(MMAX)	8F10.0



# Printout C-1. Computer printout of the tilted reflector program

```

C      TILTED REFLECTOR PROGRAM
C
      DIMENSION T(200),R(200)
      DIMENSION TITLE(20)
      DIMENSION RO(200),TO(200),RE(200),TE(200),RN(200),TN(200),PN(200)
      DIMENSION TOUT(70)
      DTR=0.017453293
1  READ 1001,TITLE
      PUNCH 1001,TITLE
      PRINT 2001,TITLE
      READ 1002,XF,ZF,ZROT
      PRINT 2002,XF,ZF,ZROT
      DZ=ZROT-ZF
      CALL ATANXY(DZ,XF,BETA)
      BEDEG=BETA/DTR
      PRINT 2003,BEDEG
      SIB=SINF(BETA)
      COB=COSF(BETA)
C
C      READ IN SHAPED SUBREFLECTOR
C
      N=0
10  N=N+1
      READ 1003,T(N),R(N)
      T(N)=T(N)*DTR
      IF (T(N)-4.0)15,20,20
15  IF (N-200)10,99,99
20  NMAX=N-1
      PRINT 2004,NMAX
C
C      TRANSFORM DATA TO NEW SYSTEM
C
      CALL TRANQ(R,T,RO,TO,PN,0.0, 1.0,SIB,COB,XF,ZF,NMAX)
      DO 24 N=1,NMAX
24  TO(N)=TO(N)*COSF(PN(N))
      CALL TRANQ(R,T,RE,TE,PN,0.0,-1.0,SIB,COB,XF,ZF,NMAX)
      DO 25 N=1,NMAX
25  TE(N)=-TE(N)*COSF(PN(N))
      CALL TRANQ(R,T,RN,TN,PN,1.0, 0.0,SIB,COB,XF,ZF,NMAX)
      PRINT 2008
      DO 26 N=1,NMAX
      TOO=TO(N)/DTR
      TNN=TN(N)/DTR
      TEE=TE(N)/DTR
      TTT=T(N)/DTR
      PNN=PN(N)/DTR
26  PRINT 2009,TTT,R(N),TOO,RO(N),TEE,RE(N),TNN,RN(N),PNN
C
C      READ DESIRED THETA VALUES
C
      READ 1004,MMAX
      IF (MMAX- 75)29,29,99
29  READ 1002,(TOUT(N),N=1,MMAX)
      NLASTO=NMAX-2
      NLASTE=NMAX-2
      NLASTN=NMAX-2
      PRINT 2007
      DO 30 MM=1,MMAX
      TTO=TOUT(MM)*DTR
C
C      INTERPOLATE TO DESIRED THETA VALUES
C
      CALL QBURP(TO,RO,TTO,RRD,DRO,NLASTO,NMAX)
      CALL QBURP(TE,RE,TTO,RRE,DRE,NLASTE,NMAX)
      CALL QBURP(TN,RN,TTO,RRN,DRN,NLASTN,NMAX)
      CALL QBURP(TN,PN,TTO,PPN,DPN,NLASTN,NMAX)
      PNN=PPN/DTR
C
C      FIND FOURIER COEFFICIENTS
C

```

# Printout C-1 (contd)

```

CALL FKQSF(RRO,RRE,RRN,PPN,A0,A1,A2)
DRN=DRN+(A1*SINF(PPN)+2.0*A2*SINF(2.0*PPN))*DPN
PRINT 2005,TOUT(MM),RRO,DRO,RRE,DRE,RRN,DRN,PPN,DPN
CALL FKQSF(DRO,DRE,DRN,PPN,DA0,DA1,DA2)
PUNCH 2006,TOUT(MM),A0,A1,A2,DA0,DA1,DA2
30 CONTINUE
GO TO 1

C
99 TYPE 2010
PAUSE
GO TO 1

C
1001 FORMAT(20A4)
1002 FORMAT(8F10.0)
1003 FORMAT(20X,2F10.0)
1004 FORMAT(10I5)
2001 FORMAT(1H1,20A4)
2002 FORMAT(28H COORDINATES OF NEW ORIGIN/
*7H X=,E15.8/
*7H Z=,E15.8/
*26H POINT OF ROTATION ZROT=,E15.8)
2003 FORMAT(23H ROTATION ANGLE BETA=,F8.4,9H DEGREES)
2004 FORMAT(31H NUMBER OF INPUT POINTS NMAX=,I4)
2005 FORMAT(F10.5,3(F14.5,F12.5),2F11.5)
2006 FORMAT(7F10.4)
2007 FORMAT(1H1,47H R AND ITS PARTIAL DERIVATIVE WITH RESPECT TO,
*31H THETA AT DESIRED OUTPUT POINTS//
*20X5HPHI=0,21X7HPHI=180,21X16HPHI AS TABULATED/
*10H THETA ,3(8X,1HR,10X,7HDR/DT ),4X3HPHI,7X7HDPHI/DT)
2008 FORMAT(/52H SURFACE DATA IN INPUT AND TRANSFORMED COORDINATES//
*7X,10HINPUT DATA,11X,15HPHI= 0 DEGREES,10X,15HPHI=180 DEGREES,15X
*,16HPHI AS TABULATED/
*4(8H THETA,8X,1HR,8X),4H PHI)
2009 FORMAT(F10.5,F12.5,3(F13.5,F12.5),F11.5)
2010 FORMAT(41H INPUT DATA EXCEEDS DIMENSION STATEMENT/
*42H RELOAD DATA AND PUSH START TO TRY AGAIN)
END

SUBROUTINE QBURP(T,R,TT,RR,DR,NLAST,NMAX)
DIMENSION T(2),R(2)
DIMENSION B(4,5),A(4)

C
C THIS SUBROUTINE NUMERICALLY INTERPOLATES AND DIFFERENTIATES THE
C FUNCTION R(T) BY FITTING A CUBIC TO R(T) AT FOUR POINTS
C
C FIND NLAST SUCH THAT T(NLAST-1) AND T(NLAST) STRADDLE THE
C DESIRED VALUE TT. IT IS ASSUMED THAT THE TABULAR VALUES OF T
C DECREASE WITH INCREASING N
9 IF(T(NLAST)-TT)10,10,20
10 IF(NLAST-2)20,20,15
15 NLAST=NLAST-1
GO TO 9
20 CONTINUE

C
C SOLVE THE SYSTEM OF FOUR EQUATIONS TO DETERMINE THE COEFFICIENTS
C
C ESTABLISH MATRIX
DO 30 I=1,4
NK=NLAST+I-2
B(I,5)=R(NK)
DO 30 J=1,4
JJ=J-1
30 B(I,J)=T(NK)**JJ

C
C DIAGONALIZE SYSTEM
DO 50 K=1,3
KK=K+1
DO 50 I=KK,4
D=B(I,K)/B(K,K)
DO 50 L=K,5
50 B(I,L)=B(I,L)-D*B(K,L)

```

# Printout C-1 (contd)

```

C
C SOLVE BY BACK SUBSTITUTION
A(4)=B(4,5)/B(4,4)
DO 70 II=1,3
  I=4-II
  L=5-II
  DO 60 J=L,4
60 B(I,5)=B(I,5)-B(I,J)*A(J)
70 A(I)=B(I,5)/B(I,I)

C
C EVALUATE CUBIC AND ITS DERIVATIVE AT INTERPOLATION POINT TT
C
RR=A(1)+A(2)*TT+A(3)*TT*TT+A(4)*TT*TT*TT
DR=A(2)+2.0*A(3)*TT+3.0*A(4)*TT*TT
RETURN
END

SUBROUTINE FKQSF(O,E,FN,P,A0,A1,A2)

C
C THIS SUBROUTINE COMPUTES THE COEFFICIENTS FOR THE TRUNCATED
C FOURIER SERIES R(THETA)=A0+A1*COS(THETA)+A2*COS(2*THETA)
C
C O=R(ZERO RADIANS)
C E=R(PI RADIANS)
C FN=R(P RADIANS) WHERE P SHOULD LIE BETWEEN 1.05 AND 2.09
C
  IF(P-1.05)100,20,20
20 IF(2.09-P)100,200,200
100 A0=(O+E)/2.0
  A1=(O-E)/2.0
  A2=0.
  RETURN
200 C=COSE(P)
  C2=COSE(2.0*P)
  D=2.0*(C2-1.0)
  A0=(C+C2)*O-2.0*FN+(C2-C)*E
  A0=A0/D
  A1=(C2-1.0)*O+(-C2+1.0)*E
  A1=A1/D
  A2=(-C-1.0)*O+2.0*FN+(C-1.0)*E
  A2=A2/D
  RETURN
  END

SUBROUTINE TRANQ(R,T,RR,TT,PP,SI,CO,SIB,COB,XF,ZF,NMAX)

C
C THIS SUBROUTINE PERFORMS A COORDINATE TRANSLATION AND ROTATION
C
  DIMENSION R(2),T(2),RR(2),TT(2),PP(2)

C
C CONVERT TO RECTANGULAR COORDINATES
DO 10 N=1,NMAX
  X=R(N)*SINF(T(N))*CO
  Y=R(N)*SINF(T(N))*SI
  Z=R(N)*COSE(T(N))

C
C PERFORM TRANSLATION
X=X-XF
Z=Z-ZF

C
C PERFORM ROTATION
D=X
X=X*COB+Z*SIB
Z=-D*SIB+Z*COB

C
C CONVERT TO POLAR COORDINATES IN NEW SYSTEM
RR(N)=SQRTF(X*X+Y*Y+Z*Z)
D=SQRTF(X*X+Y*Y)
CALL ATANXY(Z,D,TT(N))

```

# Printout C-1 (contd)

```
10 CALL ATANXY(X,Y,PP(N))  
   RETURN  
   END  
  
   SUBROUTINE ATANXY(X,Y,PHI)  
     PI=3.1415927  
     IF(X)10,20,30  
10  PHI=ATANF(Y/X)+PI  
    GO TO 40  
20  IF(Y)21,22,23  
21  PHI=-PI/2.0  
    GO TO 50  
22  PHI=0.0  
    GO TO 50  
23  PHI=PI/2.0  
    GO TO 50  
30  PHI=ATANF(Y/X)  
    GO TO 50  
40  IF(PHI-PI)50,50,41  
41  PHI=PHI-2.0*PI  
50  RETURN  
    END
```

## Appendix D

### Scattering and Spherical-Wave Expansion Programs

The scattering program consists of the following programs\* (see Fig. D-1):

MAIN	asymmetrical scattering program
FIELDS	subroutine that provides $\mathbf{H}_i$ on surface (see Section IV)
SURF	subroutine that provides $\rho$ and its derivatives (see Section III)
SETUP	subroutine that establishes integration and output grids, and tables of precomputed trigonometric functions
PATHL	subroutine that established the path length term, Eq. (17)
FINT	numerical integration subroutine (see Section V)
LEGEND	Legendre functions subroutine
SPHANK	spherical Hankel function subroutine
EXPJX	fast trigonometric subroutine (see Appendix B)
VECTOR	subroutine that converts from rectangular to polar coordinates
ADJUST	subroutine that normalizes angles to the range $-180$ to $180$ deg
PRTIM	subroutine that prints out execution times

The spherical-wave expansion program (orthogonality method) consists of the following subprograms (see Fig. D-2):

MAIN	spherical-wave expansion program (orthogonality method)
MULT	a general purpose matrix multiplication subroutine
LEGEND	Legendre function subroutine
VECTOR	subroutine that converts from rectangular to polar coordinates
PRTIM	subroutine that prints out execution times

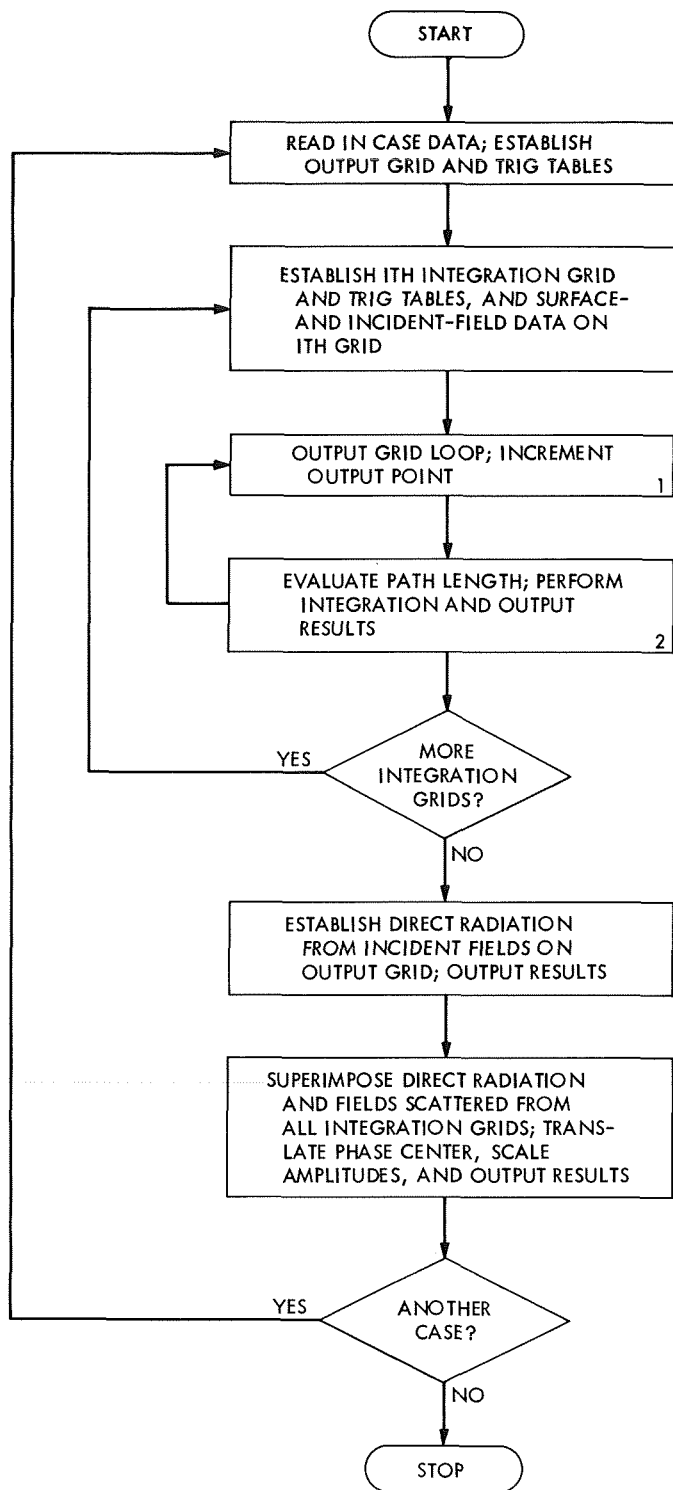
\*All the programs presented in this section, except SOLVE and PRTIM, were developed and programmed by the author. The matrix inversion package SOLVE was developed by Richard Hanson of the Computation and Analysis Section of JPL; PRTIM, which prints out the actual computation time during program execution, was developed by Charles Lawson, also of the Computation and Analysis Section.

The spherical-wave expansion program (linear equation method) consists of the following subprograms (see Fig. D-2):

MAIN	spherical-wave expansion program (linear equation method)
SOLVE	a package of subroutines to solve a set of linear equations
LEGEND	Legendre function subroutine
PRTIM	subroutine that prints out execution times

The input data required by the main scattering program are shown in Table D-1; the parameters are as follows:

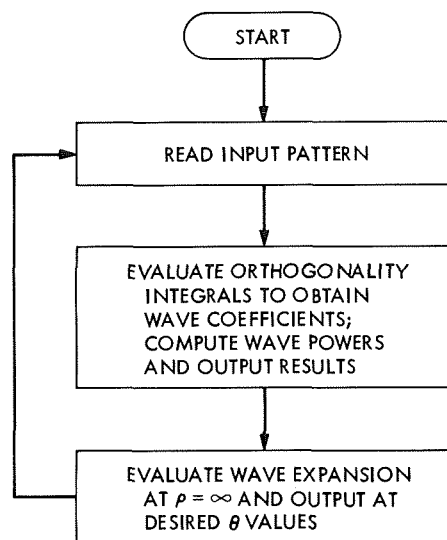
TITLE	any alphanumeric statement
PC	propagation constant, $2\pi/\lambda$
XT, YT, ZT	translations to reference point for phase pattern of output data (normally the expected phase center of the scattered pattern)
SCALE	a scale factor for the output fields; if this is left blank or if input is zero, program sets $SCALE = 1.0$
JMAX	number of $\Theta$ values desired, $JMAX \leq 181$
TT1	initial $\Theta$ value
DTT	increment for $\Theta$ values, $DTT \geq 0$
TT(1), TT(2), ...	explicit list of $\Theta$ values
KMAX	number of $\Phi$ values desired in output, $KMAX \leq 8$
PP1	initial $\Phi$ value
DPP	increment for $\Phi$ values, $DPP \geq 0$
PP(1), PP(2), ...	explicit list of $\Phi$ values
NG1	number of integration grids, $NG1 \leq 5$
MM(I)	number of $\theta$ values in the $I$ th integration grid, $MM \leq 15$
T1	initial $\theta$ value
DT	increment for $\theta$ values, $DT \geq 0$



- 1 THIS IS A  $JMAX \times KMAX$  LOOP
- 2 THIS STEP INTERNALLY CONTAINS  
SEVERAL  $MMAX \times NMAX$  LOOPS

Fig. D-1. Block diagram of scattering program

(a) ORTHOGONALITY VERSION



(b) LINEAR EQUATION VERSION

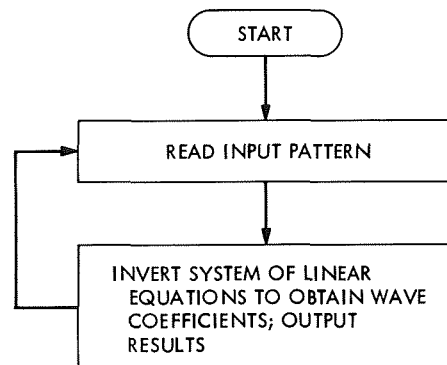


Fig. D-2. Block diagram of spherical-wave expansion program

- $T(I,1), T(I,2), \dots$  explicit list of  $\theta$  values for  $I$ th grid
- $NN(I)$  number of  $\phi$  values in the  $I$ th integration grid,  $NN \leq 91$
- $P1$  initial  $\phi$  value
- $DP$  increment for  $\phi$  values
- $P(I,1), P(I,2), \dots$  explicit list of  $\phi$  values for  $I$ th grid
- (LAST) a last case indicator; punched as shown in columns 1 through 6

In all cases, a set of angle values may be specified by an initial value and a positive increment, or by an explicit list of values (not necessarily equally spaced). All angles are in degrees. If a positive increment is given, the data cards containing explicit values *must be omitted*. If

Table D-1. Input for MAIN scattering program

Card	Parameters	Format
1	TITLE	13A6
2	PC XT YT ZT SCALE	5F10.0
3	JMAX TT1 DTT	15,2F10.0
.	TT(1) TT(2) . . .	8F10.0
.	. . . TT(JMAX)	8F10.0
.	KMAX PP1 DPP	15,2F10.0
.	PP(1) PP(2) . . .	8F10.0
.	. . . PP(KMAX)	8F10.0
.	NG1	I5
.	MM(1) T1 DT	15,2F10.0
.	T(1,1) T(1,2) . . .	8F10.0
.	. . . T(1,MM(1))	8F10.0
.	NN(1) P1 DP	15,2F10.0
.	P(1,1) P(1,2) . . .	8F10.0
.	. . . P(1,NN(1))	8F10.0
.	.	.
.	.	.
.	MM(NG1) T1 DT	15,2F10.0
.	T(NG1,1) T(NG1,2) . . .	8F10.0
.	. . . T(NG1,MM(NG1))	8F10.0
.	NN(NG1) P1 DP	15,2F10.0
.	P(NG1,1) P(NG1,2) . . .	8F10.0
.	. . . P(NG1,NN(NG1))	8F10.0
.	[Input for FIELDS Subroutine]	—
.	[Input for SURF Subroutine]	—
.	TITLE	.
.	PC XT YT ZT SCALE	.
.	.	.
.	.	.
.	(Data for further cases, same as above)	.
.	.	.
.	.	.
(Last)	(LAST)	13A6

the data cards containing explicit values are included, the increment must be written in as identically zero.

The input data required by the spherical-wave version FIELDS subroutine are shown in Table D-2; the parameters are as follows:

TITLE any alphanumeric statement  
 LMAX maximum mode order,  $\leq 60$   
 MCOMP order of azimuthal variation

Table D-2. Input for spherical-wave FIELDS subroutine

Card	Parameters	Format
1	TITLE	13A6
2	TITLE	13A6
3	LMAX MCOMP	2I5
4	1 A(1,1) A(1,2) B(1,1) B(1,2)	15,2E17.8, 2X,2E17.8
.	.	.
.	.	.
.	.	.
LMAX + 2	LMAX A(LMAX,1) A(LMAX,2) B(LMAX,1) B(LMAX,2)	.

A(N,1), A(N,2) real and imaginary parts of  $TE_{MCOMP,N}$  wave coefficient (see Appendix A and following program)

B(N,1), B(N,2) real and imaginary parts of  $TM_{MCOMP,N}$  wave coefficient (see Appendix A and following program)

The input data for the far-field version FIELDS subroutine are shown in Table D-3; the parameters are as follows:

TITLE any alphanumeric statement

JMAX not used

JO not used

JIN number of input points,  $JIN \leq 181$

IC1  $\leq 0$  for input in dB;  $> 0$  for input in V

IC2  $> 0$  to set phase pattern zero everywhere

MCOMP order of azimuthal variation

PSI polar angle  $\theta$  (see Fig. 1)

E  $|E_\theta(\theta, 90 \text{ deg})|$ , in V or dB

EP argument  $\{E_\theta(\theta, 90 \text{ deg})\}$ , in deg } see  
 H  $|E_\phi(\theta, 0 \text{ deg})|$ , in V or dB } Eq. (14)

HP argument  $\{E_\phi(\theta, 0 \text{ deg})\}$ , in deg

The input data for the SURF subroutine are shown in Table D-4; the parameters are as follows:

TITLE any alphanumeric statement

TIN input  $\theta$  value (see note below)

**Table D-3. Input for far-field FIELDS subroutine**

Card	Parameters	Format
1	TITLE	72H
2	JMAX JO JIN IC1 IC2 MCOMP	615
3	PSI(1) E(1) EP(1) H(1) HP(1)	5F10.6
.	.	.
.	.	.
JIN + 2	PSI(JIN) E(JIN) EP(JIN) H(JIN) HP(JIN)	5F10.6

A0, A1, A2  $a_0, a_1, a_2$  (see Eqs. 12 and 13, and Appendix C)

DA0, DA1, DA2  $\frac{\partial a_0}{\partial \theta}, \frac{\partial a_1}{\partial \theta}, \frac{\partial a_2}{\partial \theta}$  (see Eqs. 12 and 13, and Appendix C)

The TIN values must agree with the specified integration grid values. (If they disagree, an error message is printed but the program continues to run.) The values for each integration grid are stacked in order.

Computer time is nearly proportional to the quantity

$$P = \sum_{\text{NGI integration grids}} \text{NMAX} \times \text{MMAX}$$

For an IBM 7094, the time can vary by a factor of 2 depending on which model is being used; time for a slower model is given by

$$t = P \times [\text{JMAX} \times \text{KMAX}] 8.4 \times 10^{-5} \text{ min}$$

plus

$$t = P \times [\text{LMAX}]^2 5 \times 10^{-6} \text{ min}$$

if the spherical-wave version FIELDS subroutine is used.

The input data required by the spherical-wave expansion program (orthogonality version) are shown in Table D-5. All of these parameters have been previously defined, except for JMAXO =  $180/\Delta\theta + 1$ , where  $\Delta\theta$  is the desired output increment, and JOUT is the number of output values starting with  $\theta = 0$ . The program evaluates the spherical-wave expansion at  $\rho = \infty$  and prints and

**Table D-4. Input for SURF subroutine**

Card	Parameters	Format
1	TITLE	12A6
2	TIN(1) A0(1) A1(1) A2(1) DA0(1)	7F10.4
.	DA1(1) DA2(1)	.
.	TIN(2) A0(2) A1(2) A2(2) DA0(2)	.
.	DA1(2) DA2(2)	.
.	.	.
.	.	.
.	TIN(MM) A0(MM) A1(MM) A2(MM)	7F10.4
.	DA0(MM) DA1(MM) DA2(MM)	
.	TIN(1) A0(1) A1(1) . . .	
.	.	
.	.	

punches the results at a set of output values determined by these parameters.

The input data required by the spherical-wave expansion program (linear equation version) are identical except that the first two cards and the last card are omitted.

Both of these programs use less than 1 min of computer time to evaluate 60 TE and 60 TM wave coefficients.

Two computer printouts are included in this appendix: the first (Printout D-1) is a computer printout of the asymmetrical scattering program; the second (Printout D-2) is a printout of the spherical-wave expansion program.

**Table D-5. Input for spherical-wave expansion program**

Card	Parameters	Format
1	TITLE	13A6
2	MCOMP LMAX	215
3	TITLE	
4	JMAX JO JIN IC1 IC2 MCOMP	615
5	PSI(1) E(1) EP(1) H(1) HP(1)	5F10.0
.	.	.
.	.	.
.	.	.
JIN + 4	PSI(JIN) E(JIN) EP(JIN) H(JIN) HP(JIN)	5F10.0
JIN + 5	JMAXO JOUT	215



# Printout D-1. Computer printout of asymmetrical scattering program

## (a) Main program

```

$IBFTC MAIN
C   ASYMMETRICAL SCATTERING PROGRAM
C   PROGRAM COMPUTES SCATTERING FOR PERFECTLY CONDUCTING SMOOTH
C   SURFACE OF ARBITRARY SHAPE A. LUDWIG 1-15-69
C
C
C
C
COMMON/GRID1/SIT(15),COT(15),SIP(181),COP(181),T(15),P(181)
COMMON/GRID2/SITT(181),COTT(181),SIPP( 8),COPP( 8),TT(181),PP(8)
DIMENSION TITLE(13)
DIMENSION F( 15, 91),FT( 15, 91),FP( 15, 91),GAM( 15, 91)
INTEGER TITLE
COMPLEX HR( 15, 91),HT( 15, 91),HP( 15, 91)
COMPLEX ETT(181, 8),EPP(181, 8)
COMPLEX ETTO,EPP0
COMPLEX A( 15, 91,3)
COMPLEX STOT(3)
COMPLEX T1,T2,T3
COMPLEX TET,TEP
EQUIVALENCE (A(1,1,1),HR),(A(1,1,2),HT),(A(1,1,3),HP)
EQUIVALENCE (GAM,FT)
DATA LC/6H(LAST)/
C   START TIMING ROUTINE
CALL PRTIM1
READ(5,1001)TITLE
1  WRITE(6,2001)TITLE
  READ(5,1002)PC,XT,YT,ZT,SCALE
  IF(SCALE)22,21,22
21 SCALE=1.0
22 CONTINUE
  WRITE(6,2003)PC
C
C   READ IN GRID DATA AND ESTABLISH OUTPUT GRID AND FIRST INTEGRATION
C   GRID
CALL SETUP(NG1,I,JMAX,KMAX,MMAX,NMAX)
CALL PRTIM2
WRITE(6,2002)TITLE
C
C   READ IN EXCITATION FIELD DATA
CALL FIELDS
CALL PRTIM2
C
C   BEGIN LOOP FOR INTEGRATION GRID SEGMENTS
DO 90 J=1,JMAX
DO 90 K=1,KMAX
  ETT(J,K)=(0.0,0.0)
  90 EPP(J,K)=(0.0,0.0)
100 WRITE(6,2002)TITLE
C
C   ESTABLISH SURFACE PARAMETERS ON INTEGRATION GRID
CALL SURF(MMAX,NMAX,F,FT,FP)
DO 410 M=1,MMAX
DO 410 N=1,NMAX
  410 F(M,N)=PC*F(M,N)
C   ESTABLISH EXCITATION FIELDS ON INTEGRATION GRID
CALL FIELDI(MMAX,NMAX,HR,HT,HP,F)
C   COMBINE SURFACE AND FIELD DATA TO DETERMINE COMPLEX VECTOR A
DO 400 M=1,MMAX
DO 400 N=1,NMAX
  T1=FT(M,N)*SIT(M)*HP(M,N)-FP(M,N)*HT(M,N)
  T2=FP(M,N)*HR(M,N)+F(M,N)*SIT(M)*HP(M,N)/PC
  T3=-FT(M,N)*SIT(M)*HR(M,N)-F(M,N)*SIT(M)*HT(M,N)/PC
  A(M,N,1)=T1*SIT(M)*COP(N)+T2*COT(M)*COP(N)-T3*SIP(N)
  A(M,N,2)=T1*SIT(M)*SIP(N)+T2*COT(M)*SIP(N)+T3*COP(N)
  A(M,N,3)=T1*COT(M)-T2*SIT(M)
  400 CONTINUE
  WRITE(6,2007)I
  CALL PRTIM2
C
C   BEGIN OUTPUT GRID LOOP

```

# Printout D-1 (contd)

```

WRITE(6,2006)I
C
DO 500 K=1,KMAX
TO=PP(K)/0.017453293
WRITE(6,2011)TO
DO 500 J=1,JMAX
C
ESTABLISH PHASE/PATH LENGTH PARAMETER ON INTEGRATION GRID
CALL PATHL(F,J,K,MMAX,NMAX,GAM)
C
PERFORM INTEGRATION
CALL FINT(T,P,A,GAM,MMAX-1,NMAX-1,STOT)
C
CONVERT RESULT TO POLAR COORDINATES AND SUPERINPOSE FIELDS
C
ETTO=COTT(J)*(STOT(1)*COPP(K)+STOT(2)*SIPP(K))-STOT(3)*SITT(J)
EPPD=STOT(2)*COPP(K)-STOT(1)*SIPP(K)
ETTO=-(0.0,1.0)*PC/6.2831854*ETTO
EPPD=-(0.0,1.0)*PC/6.2831854*EPPD
TO=TT(J)/0.017453293
A1=REAL(ETTO)
A2=AIMAG(ETTO)
A3=REAL(EPPD)
A4=AIMAG(EPPD)
CALL VECTOR(A1,A2,ETAMP,ETPHI)
CALL VECTOR(A3,A4,EPAMP,EPPHI)
WRITE(6,2012)TO,ETAMP,ETPHI,EPAMP,EPPHI
ETT(J,K)=ETT(J,K)+ETTO
EPP(J,K)=EPP(J,K)+EPPD
500 CONTINUE
CALL PRTIM2
C
C
IF MORE INTEGRATION GRIDS REMAIN LOOP BACK
C
IF(I-NG1)600,700,700
600 CALL RESET(I,MMAX,NMAX)
GO TO 100
700 I=I+1
C
ESTABLISH DIRECT RADIATION ON OUTPUT GRID
WRITE(6,2009)
DO 750 K=1,KMAX
TO=PP(K)/0.017453293
WRITE(6,2011)TO
DO 750 J=1,JMAX
CALL FIELD2(J,K,ETTO,EPPD)
TO=TT(J)/0.017453293
A1=REAL(ETTO)
A2=AIMAG(ETTO)
A3=REAL(EPPD)
A4=AIMAG(EPPD)
CALL VECTOR(A1,A2,ETAMP,ETPHI)
CALL VECTOR(A3,A4,EPAMP,EPPHI)
WRITE(6,2012)TO,ETAMP,ETPHI,EPAMP,EPPHI
ETT(J,K)=ETT(J,K)+ETTO
EPP(J,K)=EPP(J,K)+EPPD
750 CONTINUE
CALL PRTIM2
C
C
TRANSLATE PHASE CENTER, SCALE AMPLITUDES, AND OUTPUT TOTAL FIELDS
C
PUNCH 1001,TITLE
WRITE(6,2002)TITLE
WRITE(6,2010)XT,YT,ZT,SCALE
DO 760 K=1,KMAX
TO=PP(K)/0.017453293
WRITE(6,2011)TO
DO 760 J=1,JMAX
TO=TT(J)/0.017453293
A1=REAL(ETT(J,K))

```

# Printout D-1 (contd)

```

A2=AIMAG(ETT(J,K))
A3=REAL(EPP(J,K))
A4=AIMAG(EPP(J,K))
CALL VECTOR(A1,A2,ETAMP,ETPHI)
CALL VECTOR(A3,A4,EPAMP,EPPHI)
DP=XT*SITT(J)*COPP(K)+YT*SITT(J)*SIPP(K)+ZT*(COTT(J)+1.0)
DP=DP*PC*57.29578
ETPHI=ETPHI-DP
EPPHI=EPPHI-DP
CALL ADJUST(ETPHI)
CALL ADJUST(EPPHI)
ETAMP=ETAMP*SCALE
EPAMP=EPAMP*SCALE
WRITE(6,2012) TO,ETAMP,ETPHI,EPAMP,EPPHI
PUNCH 2013,TO,ETAMP,ETPHI,EPAMP,EPPHI
760 CONTINUE
CALL PRTIM2
C
C   TEST FOR ANOTHER DATA BLOCK
READ(5,1001)TITLE
IF(TITLE(1)-LC)1,800,1
800 CONTINUE
WRITE(6,2004)
STOP
1001 FORMAT(13A6)
1002 FORMAT(10F10.0)
2001 FORMAT(32H1      LUDWIG SCATTERING PROGRAM//5X,13A6//)
2002 FORMAT(5H1      ,13A6//)
2003 FORMAT(26H      PROPAGATION CONSTANT=,E14.8//)
2004 FORMAT(18H1 END OF LAST CASE)
2006 FORMAT(29H0 SCATTERED FIELDS FROM GRID,I2)
2007 FORMAT(30H1 BEGIN INTEGRATION OVER GRID,I2)
2009 FORMAT(40H1 DIRECT RADIATION FROM INCIDENT FIELDS)
2010 FORMAT(63H SUPERPOSITION OF ALL GRID SCATTERED FIELDS AND DIRECT
1 FIELDS/32H PHASE CENTER TRANSLATED BY X=,F10.4,5H Y=,F10.4,
25H Z=,F10.4/
339H AMPLITUDE VALUES SCALED BY FACTOR OF,E15.8)
2011 FORMAT(7H0 PHI=,F7.2/
117X,7HE THETA,15X,5HE PHI/
29H THETA,2(20H VOLTS PHASE ))
2012 FORMAT(F9.2,F11.6,F8.2,F12.6,F8.2)
2013 FORMAT(F10.2,F10.6,F10.2,F10.6,F10.2)
END

```

## (b) Subroutine FIELDS (spherical-wave expansion)

```

$IBFTC FIELD
SUBROUTINE FIELDS
C
C   THIS SUBROUTINE EVALUATES A SPHERICAL WAVE EXPANSION TO OBTAIN
C   THE MAGNETIC FIELDS ON A SURFACE S. WAVE COEFFICIENTS ARE INPUT
C   ON CARDS. A. LUDWIG 1-14-69
C
COMMON/GRID1/SIT(15),COT(15),SIP(181),COP(181),T(15),P(181)
COMMON/GRID2/SITT(181),COTT(181),SIPP(8),COPP(8),TT(181),PP(8)
DIMENSION F(101),G(100)
DIMENSION NAME(13)
DIMENSION A(100,2),B(100,2)
C
C   READ IN WAVE COEFFICIENTS
C
READ(5,1001)NAME
WRITE(6,2001)NAME
READ(5,1001)NAME
READ(5,1002)LMAX,MCOMP
READ(5,1003)(J,A(J,1),A(J,2),B(J,1),B(J,2),L=1,LMAX)
FMC=MCOMP
WRITE(6,2003)MCOMP
WRITE(6,2002)NAME
WRITE(6,2004)(J,A(J,1),A(J,2),B(J,1),B(J,2),J=1,LMAX)
RETURN

```

# Printout D-1 (contd)

```

C
C   ENTRY POINT FOR MAGNETIC FIELDS AT FINITE R
C
    ENTRY FIELD1(MMAX,NMAX,HR,HT,HP,R)
    DIMENSION R( 15, 91)
    COMPLEX HR( 15, 91),HT( 15, 91),HP( 15, 91)
    IENT=1
    M=0
  1 M=M+1
    SN=SIT(M)
    Z=COT(M)
    TOUT=T(M)
    GO TO 99
C
C   ENTRY POINT FOR ELECTRIC FIELDS AT INFINITE R
C
    ENTRY FIELD2(JO,KO,ETTO,EPP0)
    COMPLEX ETTO,EPP0
    IENT=2
    SN=SITT(JO)
    Z=COTT(JO)
    TOUT=TT(JO)
C
C   ESTABLISH LEGENDRE FUNCTION TABLES FOR M-TH THETA VALUE
C
  99 IF(ABS(SN)-.00001)200,100,100
 100 DO 105 N=1,MCOMP
 105 F(N)=0
    NC=LMAX+1
    CALL LEGEND(NC,MCOMP,Z,F)
    DO 110 N=1,LMAX
      T1=N-MCOMP+1
      T2=N+1
      G(N)=T1*F(N+1)-T2*Z*F(N)
 110 G(N)=G(N)/SN
    DO 115 N=1,LMAX
 115 F(N)=F(N)/SN
C   F(L) IS MULTIPLIED BY FMC LATER
    GO TO (300,500),IENT
C   SPECIAL EQUATIONS FOR TH=0 AND TH=180 DEG
 200 IF(MCOMP-1)210,220,210
 210 DO 215 N=1,LMAX
    F(N)=0
 215 G(N)=0
    GO TO (300,500),IENT
 220 DO 225 N=1,LMAX
    FN=N*(N+1)
    F(N)=FN/2.0
 225 G(N)=FN/2.0
    IF(TOUT-1.57)250,250,230
 230 DO 235 N=1,LMAX,2
    F(N+1)=-F(N+1)
 235 G(N)=-G(N)
 250 GO TO (300,500),IENT
C
C   FOR EACH PHI VALUE THE HANKEL FUNCTIONS ARE EVALUATED AT THE
C   CORRESPONDING VALUE FOR R AND THE WAVE EXPANSION IS SUMMED
C
 300 DO 450 N=1,NMAX
    HRR=0
    HRI=0
    HTR=0
    HTI=0
    HPR=0
    HPI=0
    CALL SPHANK(1,R(M,N),SOR,SOI)
    DO 400 L=1,LMAX
      FL=L
      FN=(FL+1.0)/R(M,N)
      NC=L+1
      CALL SPHANK(NC,R(M,N),S1R,S1I)

```

Printout D-1 (contd)

```

F1R=-A(L,1)*S1R+A(L,2)*S1I
F1I=-A(L,1)*S1I-A(L,2)*S1R
F2R=FN*(A(L,1)*S0I+A(L,2)*SOR)
F2I=FN*(A(L,2)*S0I-A(L,1)*SOR)
F3R=-B(L,1)*SOR+B(L,2)*S0I
F3I=-B(L,1)*S0I-B(L,2)*SOR
HRR=HRR+F2R*(L)*FL
HRI=HRI+F2I*(L)*FL
F(L)=F(L)*FMC
HTR=HTR+F1R*(L)+F2R*(L)+F3R*(L)
HTI=HTI+F1I*(L)+F2I*(L)+F3I*(L)
HPR=HPR-F1R*(L)-F2R*(L)-F3R*(L)
HPI=HPI-F1I*(L)-F2I*(L)-F3I*(L)
SOR=S1R
S0I=S1I
400 CONTINUE
HRR=HRR*SN*COS(FMC*P(N))
HRI=HRI*SN*COS(FMC*P(N))
HR(M,N)=CMPLX(HRR,HRI)
HTR=HTR*COS(FMC*P(N))
HTI=HTI*COS(FMC*P(N))
HT(M,N)=CMPLX(HTR,HTI)
HPR=HPR*SIN(FMC*P(N))
HPI=HPI*SIN(FMC*P(N))
HP(M,N)=CMPLX(HPR,HPI)
450 CONTINUE
IF(M-MMAX)1,460,460
460 RETURN
500 CONTINUE
ETR=0.0
ETI=0
EPR=0
EPI=0
DO 550 L=1,LMAX
F(L)=F(L)*FMC
ETR=ETR+A(L,1)*F(L)+B(L,1)*G(L)
ETI=ETI+A(L,2)*F(L)+B(L,2)*G(L)
EPR=EPR+A(L,1)*G(L)+B(L,1)*F(L)
EPI=EPI+A(L,2)*G(L)+B(L,2)*F(L)
550 CONTINUE
ETR=ETR*SIN(FMC*PP(KO))
ETI=ETI*SIN(FMC*PP(KO))
ETTO=CMPLX(ETR,ETI)
EPR=EPR*COS(FMC*PP(KO))
EPI=EPI*COS(FMC*PP(KO))
EPO=CMPLX(EPR,EPI)
RETURN
1001 FORMAT(13A6)
1002 FORMAT(5I5)
1003 FORMAT(15,2E17.8,2X,2E17.8)
2001 FORMAT(1H,62H FIELD DATA INPUT IN THE FORM OF SPHERICAL WAVE CO
1EFFICIENTS/5X,13A6)
2002 FORMAT(5X,13A6///20X,4HA(N),32X,4HB(N)/
15H N,7X4HREAL,13X,4HIMAG,15X4HREAL,13X,4HIMAG)
2003 FORMAT(25H AZMUTHAL ORDER MCOMP=,I2)
2004 FORMAT(15,2E17.8,2X,2E17.8)
END

```

# Printout D-1 (contd)

## (c) Subroutine FIELDS (far-field approximation)

```

$IBFTC FIELD
SUBROUTINE FIELDS
C
C THIS SUBROUTINE ASSUMES FAR-FIELD BEHAVIOR FOR THE INCIDENT FIELDS
C TO OBTAIN THE MAGNETIC FIELDS ON A SURFACE S
C A. LUDWIG MARCH 1969
C
C DIMENSION TITLE(13),PSI(181),E(181),EP(181),H(181),HP(181)
C DIMENSION DUM(2)
C COMPLEX EE,EXPJX
C EQUIVALENCE (EE,DUM(1),CO),(DUM(2),SI)
C COMMON/GRID1/SIT( 15),COT( 15),SIP(181),COP(181),T( 15),P(181)
C COMMON/GRID2/SITT(181),COTT(181),SIPP( 8),COPP( 8),TT(181),PP(8)
C DTR=0.017453293
C WRITE(6,2001)
2001 FORMAT(52H ILLUMINATION FIELD FROM CIRCULARLY SYMMETRIC FEED//)
C
C READ IN INPUT FIELDS
C READ(5,1001)TITLE
C READ(5,1002)JINM,J0,JIN,IC1,IC2,MCOMP
C FMC=MCOMP
C READ(5,1003)(PSI(J),E(J),EP(J),H(J),HP(J),J=1,JIN)
C
C FOR IC1 LESS THAN OR EQUAL TO 0 CONVERT FROM DB TO VOLTS
C IF(IC1)10,10,20
10 DO 15 J=1,JIN
C E(J)=10.0**((E(J)/20.0)
15 H(J)=10.0**((H(J)/20.0)
20 CONTINUE
C
C FOR IC2 GREATER THAN ZERO NEGLECT FEED PHASE PATTERN
C IF(IC2)40,40,30
30 DO 35 J=1,JIN
C EP(J)=0
35 HP(J)=0
40 CONTINUE
C
C PRINT OUT FIELD ILLUMINATION
C WRITE(6,2002)TITLE
C WRITE(6,2003)(PSI(J),E(J),EP(J),H(J),HP(J),J=1,JIN)
C
C CONVERT POLAR ANGLES TO RADIANS
C DO 45 J=1,JIN
45 PSI(J)=DTR*PSI(J)
C
C CONVERT TO REAL AND IMAGINARY
C IF(IC2)50,50,60
50 DO 55 J=1,JIN
C TH=DTR*EP(J)
C EE=EXPJX(TH)
C EP(J)=E(J)*SI
C E(J)=E(J)*CO
C TH=DTR*HP(J)
C EE=EXPJX(TH)
C HP(J)=H(J)*SI
55 H(J)=H(J)*CO
60 CONTINUE
C
C SET FIELDS OUTSIDE OF INPUT RANGE EQUAL TO ZERO
C IF(JINM-JIN-1)72,71,70
70 PSI(JIN+2)=3.142
C E(JIN+2)=0
C EP(JIN+2)=0
C H(JIN+2)=0
C HP(JIN+2)=0
71 PSI(JIN+1)=PSI(JIN)+.01
C E(JIN+1)=0
C EP(JIN+1)=0
C H(JIN+1)=0
C HP(JIN+1)=0

```

# Printout D-1 (contd)

```

72 CONTINUE
RETURN
C
C   ENTER HERE FOR GRID1 FIELDS
C
ENTRY FIELD1(MMAX,NMAX,HR,HT,HF,R)
DIMENSION R(15,91)
COMPLEX HR(15,91),HT(15,91),HF(15,91)
C   LINEARLY INTERPOLATE FIELDS TO INTEGRATION GRID POLAR ANGLES
J=1
DO 200 M=1,MMAX
109 IF(PSI(J)-T(M))110,110,120
110 J=J+1
GO TO 109
120 F=(T(M)-PSI(J-1))/(PSI(J)-PSI(J-1))
OF=1.0-F
ETR =F*E(J)+OF*E(J-1)
ETI =F*EP(J)+OF*EP(J-1)
EPR =F*H(J)+OF*H(J-1)
100 EPI =F*HP(J)+OF*HP(J-1)
DO 200 N=1,NMAX
T1=ETR *SIN(FMC*P(N))
T2=ETI *SIN(FMC*P(N))
T3=EPR *COS(FMC*P(N))
T4=EPI *COS(FMC*P(N))
C   ASSUME FAR FIELD BEHAVIOR HR=0,HP=ET,HT=-EP
HR(M,N)=(0.0,0.0)
HT(M,N)=CMPLX(-T3,-T4)
HF(M,N)=CMPLX(T1,T2)
200 CONTINUE
RETURN
C
C   ENTER HERE FOR GRID2 FIELDS
ENTRY FIELD2(JO,KO,ETTO,EPP0)
COMPLEX ETTO,EPP0
C
C   LINEARLY INTERPOLATE FIELDS TO OUTPUT GRID POLAR ANGLES
J=1
JJ=JO
509 IF(PSI(J)-TT(JJ))510,510,520
510 IF(J.EQ.181)GO TO 520
J=J+1
GO TO 509
520 F=(TT(JJ)-PSI(J-1))/(PSI(J)-PSI(J-1))
OF=1.0-F
ETR =F*E(J)+OF*E(J-1)
ETI =F*EP(J)+OF*EP(J-1)
EPR =F*H(J)+OF*H(J-1)
500 EPI =F*HP(J)+OF*HP(J-1)
C
K=KO
T1=ETR *SIN(FMC*PP(K))
T2=ETI *SIN(FMC*PP(K))
T3=EPR *COS(FMC*PP(K))
T4=EPI *COS(FMC*PP(K))
ETTO=CMPLX(T1,T2)
EPP0=CMPLX(T3,T4)
RETURN
1001 FORMAT(13A6)
1002 FORMAT(10I5)
1003 FORMAT(5F10.6)
2002 FORMAT(13A6/47H          POLAR          E-PLANE          H-PLANE/
150H      ANGLE          VOLTS          DEG          VOLTS          DEG)
2003 FORMAT(F10.2,F12.6,F8.2,F13.6,F8.2)
END

```

## Printout D-1 (contd)

### (d) Subroutine SURF

```

$IBFTC SURFDK
  SUBROUTINE SURF(MMAX,NMAX,F,FT,FP)
C
C   THIS SUBROUTINE PROVIDES THE MAIN PROGRAM WITH RHO AND ITS
C   PARTIAL DERIVATIVES WITH RESPECT TO THETA AND PHI
C   RHO IS DENOTED BY F IN THIS PROGRAM FOR SOME OBSCURE REASON
C
  DIMENSION TIN(15),AO(15),A1(15),A2(15),DAO(15),DA1(15),DA2(15)
  DIMENSION TITLE(12)
  COMMON/GRID1/SIT(15),COT(15),SIP(181),COP(181),T(15),P(181)
  DIMENSION F(15,91),FT(15,91),FP(15,91)
C
  READ IN FOURIER COEFFICIENTS
C
  4 READ(5,1001)TITLE
  WRITE(6,2001)TITLE
  READ(5,1002)(TIN(M),AO(M),A1(M),A2(M),DAO(M),DA1(M),DA2(M),
  *M=1,MMAX)
  WRITE(6,2003)(TIN(M),AO(M),A1(M),A2(M),DAO(M),DA1(M),DA2(M),
  *M=1,MMAX)
C
  CHECK THAT INPUT DATA AND INTEGRATION GRID AGREE
C
  DO 10 M=1,MMAX
    TCHE=T(M)/0.017453293
    IF(ABS(TCHE-TIN(M))-.000001)10,99,99
  99 WRITE(6,2002)TIN(M),TCHE
  10 CONTINUE
C
  C   FILL OUT SURFACE TABLE
C
  DO 50 N=1,NMAX
    DO 50 M=1,MMAX
      COP2P=1.0-2.0*SIP(N)*SIP(N)
      F(M,N)=AO(M)+A1(M)*COP(N)+A2(M)*COP2P
      FT(M,N)=DAO(M)+DA1(M)*COP(N)+DA2(M)*COP2P
      SIP2P=2.0*SIP(N)*COP(N)
      FP(M,N)= -A1(M)*SIP(N)-2.0*A2(M)*SIP2P
    50 CONTINUE
  RETURN
  1001 FORMAT(12A6)
  1002 FORMAT(7F10.4)
  2001 FORMAT(44H0 SURFACE DATA INPUT IN THE FORM OF FOURIER/
  *37H COEFFICIENTS AND THEIR DERIVATIVES//12A6//
  *44H THETA AO(THETA) A1(THETA) A2(THETA),
  *37H DAO/DTHETA DA1/DTHETA DA2/DTHETA)
  2002 FORMAT(55H SURFACE INPUT DATA INCONSISTANT WITH INTEGRATION GRID/
  *14H INPUT ANGLE,F7.2,18H DEG GRID ANGLE,F7.2,4H DEG)
  2003 FORMAT(F8.2,6F12.4)
  END

```

### (e) Subroutine SETUP

```

$IBFTC SETD
  SUBROUTINE SETUP(NG1,I,JMAX,KMAX,MMAX,NMAX)
C
C   THIS PROGRAM ESTABLISHES THE OUTPUT GRID AND UP TO 5 INTEGRATION
C   GRIDS, AND PRECOMPUTES TRIG FUNCTIONS ON ALL OF THE GRID POINTS
C
  DIMENSION MM( 5),NN(5), T( 15, 5),P(181, 5)
  COMMON/GRID1/SIT( 15),COT( 15),SIP(181),COP(181),TG1( 15),PG1(181)
  COMMON/GRID2/SITT(181),COTT(181),SIPP( 8),COPP( 8),TT(181),PP(8)
  DIMENSION DUM(2)
  COMPLEX E,EXPJX
  EQUIVALENCE (E,DUM(1),CO),(DUM(2),SI)
  DATA L1/0606060606374/,L2/0606060604774/,L3/0606060636374/
  DATA L4/0606060474774/,L5/0341300000000/

```



# Printout D-1 (contd)

```

C
C      GRID VALUES MAY BE READ IN BY INPUTING ONLY -MAX VALUES, OR IF
C      EQUALLY SPACED VALUES ARE DESIRED, PROGRAM WILL COMPUTE VALUES
C      GIVEN A STARTING POINT AND AN INCREMENT
C
C      READ IN OUTPUT GRID
C      READ(5,1001)JMAX,TT1,DTT
C      IF(DTT)10,20,10
10  DO 15 J=1,JMAX
C      FJ=J-1
15  TT(J)=TT1+FJ*DTT
C      GO TO 30
20  READ(5,1002)(TT(J),J=1,JMAX)
30  READ(5,1001)KMAX,PP1,DPP
C      IF(DPP)40,50,40
40  DO 45 K=1,KMAX
C      FK=K-1
45  PP(K)=PP1+FK*DPP
C      GO TO 60
50  READ(5,1002)(PP(K),K=1,KMAX)
60  CONTINUE
C
C      READ IN INTEGRATION GRIDS
C      READ(5,1001)NG1
C      DO 100 I=1,NG1
C      READ(5,1001)MM(I),T1,DT
C      MMM=MM(I)
C      IF(DT)110,120,110
110 DO 115 M=1,MMM
C      FM=M-1
115 T(M,I)=T1+FM*DT
C      GO TO 130
120 READ(5,1002)(T(M,I),M=1,MMM)
130 READ(5,1001)NN(I),P1,Dp
C      NNN=NN(I)
C      IF(DP)140,150,140
140 DO 145 N=1,NNN
C      FN=N-1
145 P(N,I)=P1+FN*DP
C      GO TO 100
150 READ(5,1002)(P(N,I),N=1,NNN)
100 CONTINUE
C
C      PRINT OUT GRID DATA
C      WRITE(6,2001)
C      WRITE(6,2006)(L3,J,L5,TT(J),J=1,JMAX)
C      WRITE(6,2002)
C      WRITE(6,2006)(L4,K,L5,PP(K),K=1,KMAX)
C      WRITE(6,2004)NG1
C      DO 155 I=1,NG1
C      WRITE(6,2005)I
C      MMM=MM(I)
C      NNN=NN(I)
C      WRITE(6,2006)(L1,M,L5,T(M,I),M=1,MMM)
C      WRITE(6,2002)
155 WRITE(6,2006)(L2,N,L5,P(N,I),N=1,NNN)
C
C      ESTABLISH GRID2 TABLE
C      DTR=0.017453293
C      DO 200 J=1,JMAX
C      TT(J)=DTR*TT(J)
C      E=EXPJX(TT(J))
C      SITT(J)=SI
200 COTT(J)=CO
C      DO 210 K=1,KMAX
C      PP(K)=DTR*PP(K)
C      E=EXPJX(PP(K))
C      SIPP(K)=SI
210 COPP(K)=CO

```

# Printout D-1 (contd)

```

C
C   ESTABLISH GRID1 TABLES
C   I=0
C
C   FOR NG1 GREATER THAN 1, RE-ENTER SUBROUTINE HERE FOR NEW GRID1
C
C   ENTRY RESET(I,MMAX,NMAX)
C   I=I+1
C   MMAX=MM(I)
C   DO 400 M=1,MMAX
C     TG1(M)=DTR*T(M,I)
C     E=EXPJX(TG1(M))
C     SIT(M)=SI
400  COT(M)=CO
C     NMAX=NN(I)
C     DO 410 N=1,NMAX
C       PG1(N)=DTR*P(N,I)
C       E=EXPJX(PG1(N))
C       SIP(N)=SI
410  COP(N)=CO
C     RETURN
1001 FORMAT(I5,2F10.0)
1002 FORMAT(8F10.0)
2001 FORMAT(50H      THE FOLLOWING OUTPUT GRID HAS BEEN ESTABLISHED/
125H      ALL ANGLES IN DEGREES//)
2002 FORMAT(//)
2004 FORMAT(17H0      THE FOLLOWING,I3,40H INTEGRATION GRIDS HAVE BEEN EST
1ABLISHED//)
2005 FORMAT(18H0      SEGMENT NUMBER,I3/)
2006 FORMAT(5(A6,I3,A2,F9.4))
END

```

## (f) Subroutine PATHL

```

$IBFTC PATHLD
SUBROUTINE PATHL(RHO,J,K,MMAX,NMAX,GAM)
C   THIS SUBROUTINE EVALUATES THE PATH LENGTH FUNCTION GAMMA
C   DIMENSION RHO( 15, 91),GAM( 15, 91)
C   COMMON/GRID1/SIT( 15),COT( 15),SIP(181),CUP(181)
C   COMMON/GRID2/SITT(181),COTT(181),SIPP( 8),COPP( 8),TT(181),PP(8)
C   DO 10 M=1,MMAX
C     T1=SIT(M)*SITT(J)*COPP(K)
C     T2=SIT(M)*SITT(J)*SIPP(K)
C     T3=COT(M)*COTT(J)-1.0
C     DO 10 N=1,NMAX
10    GAM(M,N)=RHO(M,N)*(T1*COP(N)+T2*SIP(N)+T3)
C   RETURN
C   END

```

## (g) Subroutine FINT

```

$IBFTC FINTD
SUBROUTINE FINT(X,Y,F,R,MMAX,NMAX,STOT)
C
C   THIS SUBROUTINE NUMERICALLY INTEGRATES
C   THE RADIATION INTEGRAL
C   A. LUDWIG JUNE 1968
C
C   DIMENSION X( 15),Y(181),R( 15, 91)
C   DIMENSION DUM(2)
C   COMPLEX F( 15, 91,3),STOT(3),E,T1,T2,T3,A,B,C,EXPJX,      F12,F23,
1F14,SUM,TOT
C   EQUIVALENCE (E,DUM(1),CO),(DUM(2),SI)
C   DO 10 L=1,3
10  STOT(L)=(0.0,0.0)
C   DO 200 N=1,NMAX
C     DY=0.5*(Y(N+1)-Y(N))

```

# Printout D-1 (contd)

```

DO 200 M=1,MMAX
DS=DY*(X(M+1)-X(M))
R1=R(M+1,N+1)-R(M,N)
R2=R(M,N+1)-R(M+1,N)
R3=R(M,N)+R(M+1,N)
BE=(R1+R2)*0.5
CE=(R1-R2)*0.5
AL=(R3-CE)*0.5
IF (ABS(BE)-0.01) 100,100,110
100 F3I=BE*0.33333333
F1I=BE*0.5
F1R=1.0-F1I*F3I
F3R=0.5-BE*BE/8.0
GO TO 140
110 E=EXPJX(BE)
F1R=SI/BE
F1I=(1.0-CO)/BE
F3R=F1R-F1I/BE
F3I=(F1R-CO)/BE
140 IF (ABS(CE)-0.01) 150,150,160
150 F4I=CE*0.33333333
F2I=CE*0.5
F2R=1.0-F2I*F4I
F4R=0.5-CE*CE/8.0
GO TO 170
160 E=EXPJX(CE)
F2R=SI/CE
F2I=(1.0-CO)/CE
F4R=F2R-F2I/CE
F4I=(F2R-CO)/CE
170 E=EXPJX(AL)
F12=CMPLX(F1R*F2R-F1I*F2I,F1I*F2R+F1R*F2I)
F23=CMPLX(F2R*F3R-F2I*F3I,F2I*F3R+F2R*F3I)
F14=CMPLX(F1R*F4R-F1I*F4I,F1I*F4R+F1R*F4I)
DO 200 L=1,3
T1=F(M+1,N+1,L)-F(M,N,L)
T2=F(M,N+1,L)-F(M+1,N,L)
T3=F(M,N,L)+F(M+1,N,L)
B=T1+T2
C=T1-T2
A=T3-0.5*C
SUM=A*F12+B*F23+C*F14
TOT=E*SUM*DS
200 STOT(L)=STOT(L)+TOT
RETURN
END

```

## (h) Subroutine PRTIM

\$IBFTC	PRTM13	PRINT TIMERS	
	SUBROUTINE	PRTIM1	PRTM 20
C	VERSION 2,	FEBRUARY 1, 1967	PRTM 30
C	VERSION 3,	APRIL 1968.	
	TOTAL=0.		PRTM 40
	PART=0		PRTM 50
	CALL GETTIM (NTIME)		PRTM 60
	GO TO 10		PRTM 70
C			PRTM 80
C			PRTM 90
C			PRTM 100
	ENTRY PRTIM2		PRTM 110
	CALL GETTIM(ITIME)		PRTM 120
	TOTAL=FLOAT(ITIME-NTIME)/3600.		PRTM 130
	PART=FLOAT(ITIME-IPART)/3600.		PRTM 140
10	TOTSEC = TOTAL*60.		
	PARSEC = PART*60.		
	WRITE(6,1000)PART,PARSEC,TOTAL,TOTSEC		
	CALL GETTIM(IPART)		PRTM 160
	RETURN		PRTM 170
1000	FORMAT(15HOPARTIAL TIME =F10.4,5H MINS,3X,1H=,F12.4,5H SECS,		
	* 10X,12HTOTAL TIME =F10.4,5H MINS,3X,1H=,F12.4,5H SECS)		
	END		PRTM 200

**Printout D-1 (contd)**

\$IBMAP	FSTIME		FETCH AND STORE LOC 5	FSTI	10
*	FSTIME		FETCH AND STORE LOC 5	FSTI	20
*				FSTI	30
*			CALL SETTIM(ETIME)	FSTI	40
*			CALL GETTIM(ETIME)	FSTI	50
*				FSTI	60
	ENTRY	SETTIM		FSTI	70
	ENTRY	GETTIM		FSTI	80
SETTIM	CLA*	3,4		FSTI	90
	STO	5		FSTI	100
	TRA	1,4		FSTI	110
GETTIM	CLA	5		FSTI	120
	STO*	3,4		FSTI	130
	TRA	1,4		FSTI	140
	END			FSTI	150

## Printout D-2. Computer printout of spherical-wave expansion program

### (a) Main program (orthogonality technique)

```

$IBFTC MAINDK
C   SPHERICAL WAVE EXPANSION PROGRAM - ORTHOGONALITY TECHNIQUE
C   THIS PROGRAM DETERMINES THE SPHERICAL WAVE COEFFICIENTS FOR THE
C   M-TH FOURIER COMPONENT OF A RADIATION PATTERN.(SEE STRATTON P 416)
C   COEFFICIENTS ARE OBTAINED FROM ORTHOGONALITY INTEGRALS
C   THE EXPANSION MATCHES THE FAR-FIELD INPUT PATTERN WITH THE FAR-
C   FIELD FORM OF SPHERICAL WAVES. ART LUDWIG 12/23/68
C
      DIMENSION NAME(13),NAMEJ(13),NDUM(13)
      DIMENSION PA(60),PB(60)
      DIMENSION T(181),E(181),EP(181),H(181),HP(181)
      DIMENSION F(60,181),G(60,181),PM(61)
      DIMENSION A(181,2),B(181,2)
      DIMENSION ACDE(60,2),BCDE(60,2)
      DIMENSION AOUT(1,2),BOUT(1,2)
      DATA NDUM/13*060606060606060/
      DTR=0.017453293
C
C   READ IN M-TH COMPONENT OF INPUT PATTERN
C
      1 READ(5,1001)NAMEJ
      CALL PRTIM1
      READ(5,1002)MCOMP,NMAX
      READ(5,1001)NAME
      READ(5,1002)JMAX,JO,JIN,IC1,IC2
      READ(5,1003)(T(M),E(M),EP(M),H(M),HP(M),M=1,JIN)
C
C   FOR IC1 LESS THAN OR EQUAL TO 0 CONVERT FROM DB TO VOLTS
      IF(IC1)10,10,20
      10 DO 15 J=1,JIN
      E(J)=10.0**(E(J)/20.0)
      15 H(J)=10.0**(H(J)/20.0)
      20 CONTINUE
C   FOR IC2 GREATER THAN 0 NEGLECT PHASE
      IF(IC2)40,40,30
      30 DO 35 J=1,JIN
      EP(J)=0.
      35 HP(J)=0.
      40 CONTINUE
C   PRINT OUT INPUT PATTERN
      WRITE(6,2001)NAMEJ
      WRITE(6,2006)MCOMP
      WRITE(6,2002)NAME
      WRITE(6,2003)(T(M),E(M),EP(M),H(M),HP(M),M=1,JIN)
C   CONVERT TO RADIANS AND REAL AND IMAGINARY
      DO 45 J=1,JIN
      45 T(J)=DTR*T(J)
      IF(IC2)50,50,60
      50 DO 55 J=1,JIN
      TH=DTR*EP(J)
      EP(J)=E(J)*SIN(TH)
      E(J)=E(J)*COS(TH)
      TH=DTR*HP(J)
      HP(J)=H(J)*SIN(TH)
      55 H(J)=H(J)*COS(TH)
      60 CONTINUE
C
C   ESTABLISH A(THETA)DTHETA AND B(THETA)DTHETA VECTORS
C
      A(1,1)=E(1)*T(1)
      A(1,2)=EP(1)*T(1)
      B(1,1)=H(1)*T(1)
      B(1,2)=HP(1)*T(1)
      DO 65 J=2,JIN
      DTH=T(J)-T(J-1)
      A(J,1)=E(J)*DTH
      A(J,2)=EP(J)*DTH
      B(J,1)=H(J)*DTH
      65 B(J,2)=HP(J)*DTH
      CALL PRTIM2

```

# Printout D-2 (contd)

```

C
C      OBTAIN F AND G MATRICES
C
      FMC=MCOMP
      DO 80 J=1,JIN
      Z=COS(T(J))
      DO 85 N=1,MCOMP
85    PM(N)=0.
      NC=NMAX+1
      CALL LEGEND(NC,MCOMP,Z,PM)
      DO 80 I=1,NMAX
      F(I,J)=PM(I)*FMC
      T1=I-MCOMP+1
      T2=I+1
      G(I,J)=T1*PM(I+1)-T2*Z*PM(I)
80    CONTINUE
C
C      PERFORM NUMERICAL INTEGRATION BY MATRIX MULTIPLICATION
C
      CALL MULT(JIN,NMAX,2,F,A,ACOE,0,60,181,181,2,60,2)
      CALL MULT(JIN,NMAX,2,G,B,ACOE,1,60,181,181,2,60,2)
      CALL MULT(JIN,NMAX,2,F,B,BCOE,0,60,181,181,2,60,2)
      CALL MULT(JIN,NMAX,2,G,A,BCOE,1,60,181,181,2,60,2)
C
C      NORMALIZE COEFFICIENTS AND COMPUTE POWERS
C
      PIO2Z=(3.1415927/2.0)*0.002655
      PTOT=0.
      DO 95 N=1,NMAX
      PA(N)=0.
      PB(N)=0.
      FACT=1.
      IF(MCOMP)92,92,90
90    DO 91 M=1,MCOMP
      FF=(N-M+1)*(N+M)
91    FACT=FACT*FF
92    FF=2*N+1
      FACT=FACT/FF
      FF=2*N*(N+1)
      FACT=FACT*FF
      K=0
93    K=K+1
      BT=BCOE(N,K)
      AT=ACOE(N,K)
      BCOE(N,K)=BCOE(N,K)/FACT
      ACOE(N,K)=ACOE(N,K)/FACT
      PA(N)=PA(N)+AT*ACOE(N,K)*PIO2Z
      PB(N)=PB(N)+BT*BCOE(N,K)*PIO2Z
      IF(K-1)93,93,94
94    CONTINUE
      PTOT=PTOT+PA(N)+PB(N)
95    CONTINUE
C
C      OUTPUT COEFFICIENTS
C
      WRITE(6,2001)NAMEJ
      PUNCH 1001,NAMEJ
      PUNCH 1001,NAME
      PUNCH 2010,NMAX,MCOMP
      WRITE(6,2004)MCOMP
      PSUM=0
      DO 100 J=1,NMAX
      PA(J)=PA(J)/PTOT
      PB(J)=PB(J)/PTOT
      PSUM=PSUM+PA(J)+PB(J)
      PUNCH 2005, J,ACOE(J,1),ACOE(J,2),BCOE(J,1),BCOE(J,2)
100  WRITE(6,2005)J,ACOE(J,1),ACOE(J,2),BCOE(J,1),BCOE(J,2),PA(J),PB(J)
      *,PSUM
      WRITE(6,2007)PTOT
      CALL PRTIM2

```

# Printout D-2 (contd)

```

C
C      OUTPUT SUMMATION OF SPHERICAL MODES
C
      READ(5,1002)JMAX,JIN
      JO=0
      IC1=1
      IC2=-1
      PUNCH 1001,NAMEJ
      PUNCH 1002,JMAX,JO,JIN,IC1,IC2
      WRITE(6,2008)NAMEJ
      WRITE(6,2002)NDUM
      DT=JMAX-1
      DT=180.0/DT
      J=1
      TH=0
      IF(MCUMP-1)170,180,170
170 DO 175 N=1,NMAX
      F(1,N)=0
175 G(1,N)=0
      GO TO 215
180 DO 190 N=1,NMAX
      FN=N*(N+1)
      F(1,N)=FN/2.0
190 G(1,N)=FN/2.0
      GO TO 215
200 FJ=J-1
      TH=FJ*DT
      Z=COS(TH*DTR)
      S=SIN(TH*DTR)
      DO 205 N=1,MCOMP
205 PM(N)=0
      CALL LEGEND(NC,MCOMP,Z,PM)
      DO 210 N=1,NMAX
      F(1,N)=FMC*PM(N)/S
      T1=N-MCOMP+1
      T2=N+1
      G(1,N)=T1*PM(N+1)-T2*Z*PM(N)
210 G(1,N)=G(1,N)/S
215 CALL MULT(NMAX,1,2,F,ACOE,AOUT,0, 60,181, 60,2,1,2)
      CALL MULT(NMAX,1,2,G,BCOE,AOUT,1, 60,181, 60,2,1,2)
      CALL MULT(NMAX,1,2,G,ACOE,BOUT,0, 60,181, 60,2,1,2)
      CALL MULT(NMAX,1,2,F,BCOE,BOUT,1, 60,181, 60,2,1,2)
      CALL VECTOR(AOUT(1,1),AOUT(1,2),EAMP,EPHI)
      CALL VECTOR(BOUT(1,1),BOUT(1,2),HAMP,HPHI)
      PUNCH 2009, TH,EAMP,EPHI,HAMP,HPHI
      WRITE(6,2003)TH,EAMP,EPHI,HAMP,HPHI
      J=J+1
      IF(J-JIN)200,280,300
280 IF(J-JMAX)200,285,300
285 ISIGN=1
      TH=180.
      IF(MCUMP-1)370,380,370
370 DO 375 N=1,NMAX
      F(1,N)=0
375 G(1,N)=0
      GO TO 215
380 DO 290 N=1,NMAX
      ISIGN=-ISIGN
      FN=N*(N+1)*ISIGN
      F(1,N)=-FN/2.0
290 G(1,N)=FN/2.0
      GO TO 215
300 CONTINUE
      CALL PRTIM2
      GO TO 1
1001 FORMAT(13A6)
1002 FORMAT(10I5)
1003 FORMAT(5F10.0)
2001 FORMAT(1H1,13A6)
2002 FORMAT(1H0,13A6/46H      POLAR      E-PLANE      H-PLANE/

```

## Printout D-2 (contd)

```

150H      ANGLE      VOLTS      DEG      VOLTS      DEG)
2003 FORMAT(F10.2,F12.6,F8.2,F13.6,F8.2)
2004 FORMAT(51H0      SPHERICAL WAVE COEFFICIENTS FOR AZMUTHAL ORDER,12//
120X,4HA(N),32X,4HB(N),27X,28HFRACTION OF TOTAL MODE POWER/
25H      N,7X,4HREAL,13X,4HIMAG,15X,4HREAL,13X,4HIMAG,
313X,7HA MODES,6X,7HB MODES,5X,16HCUMULATIVE TOTAL)
2005 FORMAT(15,2E17.8,2X,2E17.8,2X,2E14.5,F14.8)
2006 FORMAT(49H0      INPUT PATTERN FOR AZMUTHAL COMPONENT OF ORDER,12)
2007 FORMAT(1H0,19H      TOTAL MODE POWER,E15.8,6H WATTS)
2008 FORMAT(1H1,13A6/41H      FAR FIELD SUMMATION OF SPHERICAL MODES)
2009 FORMAT(F10.2,F10.6,F10.2,F10.6,F10.2)
2010 FORMAT(2I5)
      END

```

### (b) Main program (linear equation technique)

```

$IBFTC MAINDK
C      SPHERICAL WAVE EXPANSION PROGRAM - LINEAR EQUATION TECHNIQUE
C      THIS PROGRAM DETERMINES THE SPHERICAL WAVE COEFFICIENTS FOR THE
C      M-TH FOURIER COMPONENT OF A RADIATION PATTERN.(SEE STRATTON P 416)
C      THE COEFFICIENTS ARE OBTAINED BY INVERTING A SYSTEM OF LINEAR
C      EQUATIONS. WARNING- THE INPUT DATA POINTS SHOULD BE NEARLY EQUALLY
C      SPACED BETWEEN 0 AND 180 DEGREES. OTHERWISE THE PROBLEM BECOMES
C      VERY ILL CONDITIONED
C      THE EXPANSION MATCHES THE FAR-FIELD INPUT PATTERN WITH THE FAR-
C      FIELD FORM OF SPHERICAL WAVES. ART LUDWIG 12/23/68
C
      DIMENSION NAME(13)
      DIMENSION T(181),E(181),EP(181),H(181),HP(181)
      DIMENSION A(100,100),XR(100),XI(100),BR(100),BI(100),PM(50)
      DIMENSION C(100,100),S(500)
      DTR=0.017453293
C
C      READ IN M-TH COMPONENT OF INPUT PATTERN
C
      1 READ(5,1001)NAME
      CALL PRTIM1
      READ(5,1002)JMAX,JO,JIN,IC1,IC2,MCOMP
      READ(5,1003)(T(M),E(M),EP(M),H(M),HP(M),M=1,JIN)
C
C      FOR IC1 LESS THAN OR EQUAL TO 0 CONVERT FROM DB TO VOLTS
      IF(IC1)10,10,20
      10 DO 15 J=1,JIN
         E(J)=10.0**((E(J)-20.0)/20.0)
      15 H(J)=10.0**((H(J)-20.0)/20.0)
      20 CONTINUE
C
C      FOR IC2 GREATER THAN 0 NEGLECT PHASE
      IF(IC2)40,40,30
      30 DO 35 J=1,JIN
         EP(J)=0.
      35 HP(J)=0.
      40 CONTINUE
C
C      PRINT OUT INPUT PATTERN
      WRITE(6,2002)NAME
      WRITE(6,2003)(T(M),E(M),EP(M),H(M),HP(M),M=1,JIN)
C
C      CONVERT TO RADIAN AND REAL AND IMAGINARY
      DO 45 J=1,JIN
      45 T(J)=DTR*T(J)
         IF(IC2)50,50,60
      50 DO 55 J=1,JIN
         TH=DTR*EP(J)
         EP(J)=E(J)*SIN(TH)
         E(J)=E(J)*COS(TH)
         TH=DTR*HP(J)
         HP(J)=H(J)*SIN(TH)
      55 H(J)=H(J)*COS(TH)
      60 CONTINUE

```



# Printout D-2 (contd)

```

C
C THE COEFFICIENT VECTOR X IS OBTAINED FROM THE MATRIX EQUATION
C AX=B, WHERE THE B VECTOR IS DEFINED IN THE FOLLOWING SECTION
C BR AND BI ARE REAL AND IMAGINARY PARTS OF B
C
      DO 70 J=1,JIN
      BR(2*J-1)=E(J)
      BR(2*J)=H(J)
      BI(2*J-1)=EP(J)
70 BI(2*J)=HP(J)
C
C THE ELEMENTS OF THE MATRIX A ARE OBTAINED FROM ASSOCIATED
C LEGENDRE POLYNOMIALS AND THEIR DERIVATIVES AS FOLLOWS
C
      CALL PRTIM2
      FMC=MCOMP
      DO 80 K=1,JIN
      Z=COS(T(K))
      S=SIN(T(K))
      DO 85 N=1,MCOMP
85 PM(N)=0.
      NMAX=JIN+1
      CALL LEGEND(NMAX,MCOMP,Z,PM)
      DO 80 J=1,JIN
      F=FMC*PM(J)/S
      T1=J-MCOMP+1
      T2=J+1
      G=T1*PM(J+1)-T2*Z*PM(J)
      G=G/S
      A(2*K-1,2*J-1)=F
      A(2*K,2*J)=F
      A(2*K-1,2*J)=G
      A(2*K,2*J-1)=G
80 CONTINUE
C
C THE SYSTEM IS INVERTED TO YIELD X
C
      EPS=.00000001
      NSOLVE=2*JIN
      CALL PRTIM2
      CALL SOLVE(100,A,NSOLVE,BR,XR,1,EPS,2,IT1,C,S)
      CALL SOLVE(100,A,NSOLVE,BI,XI,2,EPS,2,IT2,C,S)
      CALL PRTIM2
      IF(IT1)99,99,90
90 IF(IT2)99,99,95
99 WRITE(6,2010) IT1,IT2
95 CONTINUE
C
C OUTPUT RESULTS
C
      PUNCH 1001,NAME
      WRITE(6,2004)
      DO 100 J=1,JIN
      PUNCH 2005, J,XR(2*J-1),XI(2*J-1),XR(2*J),XI(2*J)
100 WRITE(6,2005) J,XR(2*J-1),XI(2*J-1),XR(2*J),XI(2*J)
      GO TO 1
1001 FORMAT(13A6)
1002 FORMAT(10I5)
1003 FORMAT(5F10.0)
2002 FORMAT(1H1,13A6/46H POLAR E-PLANE H-PLANE/
150H ANGLE VOLTS DEG VOLTS DEG)
2003 FORMAT(F10.2,F12.6,F8.2,F13.6,F8.2)
2004 FORMAT(1H1,29H SPHERICAL WAVE COEFFICIENTS//
120X,4HA(N),26X,4HB(N)/
25H N,7X,4HREAL,13X,4HIMAG,15X,4HREAL,13X,4HIMAG/)
2005 FORMAT(15,2E17.8,2X,2E17.8)
2010 FORMAT(47H CAUTION-FULL ACCURACY NOT ACHIEVED IN MATRIX/
120H INVERSION. IT1=,I2,7H IT2=,I2)
      END

```

## Printout D-2 (contd)

### (c) Subroutine LEGEND

```
$IBFTC LEGDK FULIST,DECK
      SUBROUTINE LEGEND(NMAX,M,Z,VAL)
C
C   THIS SUBROUTINE RETURNS VALUES OF THE ASSOCIATED LEGENDRE FUNCTION
C   WITH INTEGER INDICES FROM N=M TO N=NMAX.
C   DO NOT USE NMAX LESS THAN M+1
C   VALUES ARE OBTAINED USING UPWARD RECURSION, AND HAVE CHECKED WITH
C   TABULATED VALUES TO 5 PLACES THRU N=56 FOR M=1, AND N=10 FOR M=5
C   HIGHER INDICES WERE NOT CHECKED.
C   ART LUDWIG 6/6/64
C   DIMENSION VAL(100)
      DOUBLE PRECISION TERM1,TERM2,TERM3,ZD
      ZD=Z
      FM=M
      TERM1=0.
      IF(M)8,8,9
8     TERM2=1.0
      GO TO 11
9     KMAX=2*M-1
      TERM2=(1.0-ZD*ZD)**(FM/2.0)
      DO 10 K=1,KMAX,2
      FK=K
10    TERM2=TERM2*(2.0*FM-FK)
      VAL(M)=TERM2
11    NN=NMAX-1
      DO 20 N=M,NN
      FN=N
      TERM3=((2.0*FN+1.0)*ZD*TERM2-(FN+FM)*TERM1)/(FN-FM+1.0)
      VAL(N+1)=TERM3
      TERM1=TERM2
20    TERM2=TERM3
      RETURN
      END
```

### (d) Subroutine SPHANK

```
$IBFTC SPHAND
      SUBROUTINE SPHANK(N,R,X,Y)
C
C   THE VALUE OF THE SPHERICAL HANKEL FUNCTION IS RETURNED, MULTIPLIED
C   BY THE FACTOR  $(-J)^{(N+1)} * \rho * \exp(J*\rho)$ 
C
      DOUBLE PRECISION TERM,AR,AI
      AR=0
      AI=0
      PI=3.1415927
      K=0
      TERM=1
      GO TO 100
20    K=K+1
      T1=N+K
      T2=N-K+1
      T3=2*K
      TERM=TERM*T1*T2/T3
      TERM=TERM/R
      GO TO(200,100),IGU
100   AR=AR+TERM
      IGO=1
      IF(K-N)20,1000,1000
200   AI=AI-TERM
      IGO=2
      TERM=-TERM
      IF(K-N)20,1000,1000
1000  X=AR
      Y=AI
      RETURN
      END
```

## Printout D-2 (contd)

### (e) Subroutine VECTOR

```
$IBFTC VDECK
      SUBROUTINE VECTOR(X,Y,AMP,PHI)
C      THIS SUBROUTINE CONVERTS COMPLEX DATA TO POLAR FORM
C
      C=0.
      IF(X)100,200,300
100  IF(Y)110,120,120
110  C=360.
120  PHI=ATAN(Y/X)*57.29577951+180.-C
      GO TO 400
200  IF(Y)210,220,230
210  PHI=-90.
      GO TO 400
220  PHI=0.
      GO TO 400
230  PHI=90.
      GO TO 400
300  PHI=ATAN(Y/X)*57.29577951
400  AMP=SQRT(X*X+Y*Y)
      RETURN
      END
```

### (f) Subroutine ADJUST

```
$IBFTC ADJUD
      SUBROUTINE ADJUST(PHI)
C      THIS SUBROUTINE SHIFTS PHI UNTIL IT LIES IN THE RANGE -180,180
      1 IF(PHI-180.0)20,20,10
10  PHI=PHI-360.
      GO TO 1
20  IF(PHI+180.0)30,40,40
30  PHI=PHI+360.
      GO TO 20
40  RETURN
      END
```

### (g) Subroutine MULT

```
$IBFTC MULTD FULIST
      SUBROUTINE MULT(M,N,K,A,B,C,IC,NA,MA,NB,MB,NC,MC)
C      THIS IS A GENERAL PURPOSE MATRIX MULTIPLICATION SUBROUTINE
C
      DIMENSION A(NA,MA),B(NB,MB),C(NC,MC)
      IF(IC)10,20,30
10  DO 15 I=1,N
      DO 15 J=1,K
      DO 15 L=1,M
15  C(I,J)=C(I,J)-A(I,L)*B(L,J)
      RETURN
20  DO 25 I=1,N
      DO 25 J=1,K
      C(I,J)=0.
      DO 25 L=1,M
25  C(I,J)=C(I,J)+A(I,L)*B(L,J)
      RETURN
30  DO 35 I=1,N
      DO 35 J=1,K
      DO 35 L=1,M
35  C(I,J)=C(I,J)+A(I,L)*B(L,J)
      RETURN
      END
```

# Printout D-2 (contd)

\$IBMAP	DOTSLV		DOT,SDOT,DAD, ABD ILOG2 FOR USE WITH SOLVE(SLVIT)	DOTS	10
*			17 FEB. 1968	DOTS	20
*			R. J. HANSON, JPL.	DOTS	30
*	DOT AND FRIENDS		ROUTINES FOR USE WITH SOLVE (SLVIT)	DOTS	40
	ENTRY	DOT (N,A(1),MA,B(1),MB,C)	DOUBLE INNER PRODUCT	DOTS	50
	ENTRY	SDOT (N,A(1),MA,B(1),MB,C)	INNER PRODUCT	DOTS	60
	ENTRY	ILOG2 (A)	FLOATING POINT EXPONENT	DOTS	70
	ENTRY	DAD (A,B)	ADD WITH ROUND	DOTS	80
*				DOTS	90
SNAD	MACRO	M	STD, COMPLEMENTING IF NECESSARY	DOTS	100
	CHS			DOTS	110
	ADD	=1B2		DOTS	120
	ALS	18		DOTS	130
	STD	M		DOTS	140
	ENDM	SNAD		DOTS	150
SAV	MACRO	A		DOTS	160
	SXA	A,1		DOTS	170
	SXA	A+1,2		DOTS	180
	SXA	A+2,4		DOTS	190
	ENDM	SAV,NOCRS		DOTS	200
RETUR	MACRO	A		DOTS	210
	AXT	**,1		DOTS	220
	AXT	**,2		DOTS	230
	AXT	**,4		DOTS	240
	TRA	1,4		DOTS	250
	ENDM	RETUR,NOCRS		DOTS	260
DOT	SAV	RET		DOTS	270
	STZ	S		DOTS	280
	STZ	S+1		DOTS	290
	CLA*	8,4		DOTS	300
	LDQ	C+1		DOTS	310
	STO	C		DOTS	320
	CLA*	3,4		DOTS	330
	TZE	NONE	SKIP LOOP IF N = 0	DOTS	340
	STO	N		DOTS	350
	CLA	4,4		DOTS	360
	PAC	,1	X1=-(BASE OF A)	DOTS	370
	CLA*	5,4	MA	DOTS	380
	SNAD	MA		DOTS	390
	CLA	6,4		DOTS	400
	PAC	,2	X2=-(BASE OF B)	DOTS	410
	CLA*	7,4		DOTS	420
	SNAD	MB		DOTS	430
	LXA	N,4		DOTS	440
LOOP	LDQ	0,1	A(1)	DOTS	450
	FMP	0,2	B(1)	DOTS	460
	DFAD	S		DOTS	470
	DST	S		DOTS	480
MA	TXI	*,1,1,**	(X1)=(X1)+MA	DOTS	490
MB	TXI	*,1,2,**	(X2)=(X2)+MB	DOTS	500
	TIX	LOOP,4,1	END OF MAIN LOOP	DOTS	510
NONE	DFAD	C		DOTS	520
	FRN			DOTS	530
RET	RETUR			DOTS	540
*				DOTS	550
SDOT	SAV	SRET		DOTS	560
	STZ	S		DOTS	570
	CLA*	8,4		DOTS	580
	STO	C		DOTS	590
	CLA*	3,4		DOTS	600
	TZE	SNONE		DOTS	610
	STO	N		DOTS	620
	CLA	4,4		DOTS	630
	PAC	,1		DOTS	640
	CLA*	5,4		DOTS	650
	SNAD	SMA		DOTS	660
	CLA	6,4		DOTS	670
	PAC	,2		DOTS	680
	CLA*	7,4		DOTS	690
	SNAD	SMB		DOTS	700

## Printout D-2 (contd)

	LXA	N,4	DOTS 710
SLOOP	LDQ	0,1	DOTS 720
	FMP	0,2	DOTS 730
	FAD	S	DOTS 740
	STO	S	DOTS 750
SMA	TXI	**+1,1,**	DOTS 760
SMB	TXI	**+1,2,**	DOTS 770
	TIX	SLOOP,4,1	DOTS 780
SNONE	FAD	C	DOTS 790
	FRN		DOTS 800
SRET	RETUR		DOTS 810
*			DOTS 820
ILOG2	CAL*	3,4	DOTS 830
	ANA	=03770000000000	DOTS 840
	SUB	=02000000000000	DOTS 850
	ARS	27	DOTS 860
	TRA	1,4	DOTS 870
*			DOTS 880
DAD	CLA*	3,4	DOTS 890
	FAD*	4,4	DOTS 900
	FRN		DOTS 910
	TRA	1,4	DOTS 920
*			DOTS 930
	EVEN		DOTS 940
C	PZE		DOTS 950
	PZE		DOTS 960
S	PZE		DOTS 970
	PZE		DOTS 980
N	PZE		DOTS 990
T	PZE		DOTS1000
	END		DOTS1020

## (h) Subroutine SOLVE

\$IBFTC	SLVEIT	FULIST,DECK	SLVE0010
	SUBROUTINE	SOLVE(L,A,N,B,X,IN,EPS,ITMAX,IT,AA,S)	SLVE0020
CSOLVE	LINEAR	EQUATION SOLVER WITH ITERATIVE IMPROVEMENT	SLVE0030
C	SOLVES	AX=B WHERE A IS NXN MATRIX AND B IS NX1 VECTOR	SLVE0040
C			SLVE0050
C	IN=		SLVE0060
C	1	FOR FIRST ENTRY	SLVE0070
C	2	FOR SUBSEQUENT ENTRIES WITH NEW B	SLVE0080
C	3	TO RESTORE A AND B	SLVE0090
C	4	IF FIRST ENTRY BUT AN INITIAL APPROXIMATION IS ALREADY	SLVE0100
C	AVAILABLE	IN X	SLVE0110
C	5	CONTINUE CALCULATING ITERATIVE IMPROVEMENT FOR THIS SYSTEM	SLVE0120
C			SLVE0130
C	EPS	AND ITMAX ARE PARAMETERS IN THE ITERATION	SLVE0140
C	IT=		SLVE0150
C	-1	IF A IS SINGULAR	SLVE0160
C	0	IF NOT CONVERGENT	SLVE0170
C		NUMBER OF ITERATIONS IF CONVERGENT	SLVE0180
C	CALLS	MAP SUBROUTINES ILOG2, DOT, SDOT AND DAD	SLVE0190
C			SLVE0200
	DIMENSION	A(L,L),B(L),X(L),AA(N,N),S(1)	SLVE0210
	MA=L		SLVE0220
C	MA	MUST = DECLARED DIMENSION OF SYSTEM	SLVE0230
	J1=N		SLVE0240
	J2=J1+N		SLVE0250
	J3=J2+N		SLVE0260
	GO TO	(10,230,390,10,290),IN	SLVE0270
10	NM1=N-1		SLVE0280
	NP1=N+1		SLVE0290
C			SLVE0300
C	EQUILIBRATION		SLVE0310
C			SLVE0320
	DO 40	I=1,N	SLVE0330
		KTOP=ILOG2(A(I,1))	SLVE0340
		DO 20 J=2,N	SLVE0350
20	KTOP=MAX0	(KTOP,ILOG2(A(I,J)))	SLVE0360
	N2=J2+I		SLVE0370
	S(N2)=2.0**	(-KTOP)	SLVE0380
	DO 30	J=1,N	SLVE0390

# Printout D-2 (contd)

30	A(I,J)=A(I,J)*S(N2)	SLVE0400
40	CONTINUE	SLVE0410
C		SLVE0420
C	SAVE EQUILIBRATED DATA	SLVE0430
C		SLVE0440
	DO 50 I=1,N	SLVE0450
	DO 50 J=1,N	SLVE0460
	50 AA(I,J)=A(I,J)	SLVE0470
C		SLVE0480
C	GAUSSIAN ELIMINATION WITH PARTIAL PIVOTING	SLVE0490
C		SLVE0500
	DO 170 M=1,NM1	SLVE0510
	N3=J3+M	SLVE0520
	TOP=ABS(A(M,M))	SLVE0530
	IMAX=M	SLVE0540
	DO 70 I=M,N	SLVE0550
	IF(TOP-ABS(A(I,M)))60,70,70	SLVE0560
60	TOP=ABS(A(I,M))	SLVE0570
	IMAX=I	SLVE0580
70	CONTINUE	SLVE0590
	IF(TOP)90,80,90	SLVE0600
80	IT=-1	SLVE0610
C	*SINGULAR*	SLVE0620
	RETURN	SLVE0630
90	S(N3)=IMAX	SLVE0640
100	IF(IMAX-M)130,130,110	SLVE0650
110	DO 120 J=1,N	SLVE0660
	TEMP=A(M,J)	SLVE0670
	A(M,J)=A(IMAX,J)	SLVE0680
120	A(IMAX,J)=TEMP	SLVE0690
130	MP1=M+1	SLVE0700
	DO 160 I=MP1,N	SLVE0710
	EM=A(I,M)/A(M,M)	SLVE0720
	A(I,M)=EM	SLVE0730
	IF(EM)140,160,140	SLVE0740
140	DO 150 J=MP1,N	SLVE0750
	IF(ILOG2(A(M,J)) + ILOG2(EM) .LT. -54) GO TO 150	SLVE0760
	A(I,J)=A(I,J)-A(M,J)*EM	SLVE0770
C 150	A(I,J)=A(I,J)-A(M,J)*EM	SLVE0780
150	CONTINUE	SLVE0790
160	CONTINUE	SLVE0800
170	CONTINUE	SLVE0810
	N4=N*4	SLVE0820
	S(N4)=N	SLVE0830
	IF (A(N,N))190,180,190	SLVE0840
180	IT=-1	SLVE0850
	RETURN	SLVE0860
190	CONTINUE	SLVE0870
C	STORAGE FOR A NOW CONTAINS TRIANGULAR L AND U SO THAT (L+I)*U=A	SLVE0880
C		SLVE0890
C	DUPLICATE INTERCHANGES IN DATA	SLVE0900
C		SLVE0910
	DO 220 I=1,N	SLVE0920
	N3=J3+I	SLVE0930
	IP=S(N3)	SLVE0940
	IF (I-IP)200,220,200	SLVE0950
200	DO 210 J=1,N	SLVE0960
	TEMP=AA(I,J)	SLVE0970
	AA(I,J)=AA(IP,J)	SLVE0980
210	AA(IP,J)=TEMP	SLVE0990
220	CONTINUE	SLVE1000
C		SLVE1010
C	PROCESS RIGHT HAND SIDE	SLVE1020
C		SLVE1030
230	DO 240 I=1,N	SLVE1040
	N2=J2+I	SLVE1050
240	B(I)=B(I)*S(N2)	SLVE1060
	DO 250 I=1,NM1	SLVE1070
	N3=J3+I	SLVE1080
	IP=S(N3)	SLVE1090

# Printout D-2 (contd)

TEMP=B(I)	SLVE1100
B(I)=B(IP)	SLVE1110
B(IP)=TEMP	SLVE1120
250 CONTINUE	SLVE1130
IF(IN-4) 260,290,290	SLVE1140
C	SLVE1150
C BYPASS INITIAL APPROXIMATION CALCULATION	SLVE1160
C IN GREATER THAN 4 CANNOT OCCUR AT THIS POINT	SLVE1170
C	SLVE1180
C	SLVE1190
C SOLVE FOR FIRST APPROXIMATION TO X	SLVE1200
C	SLVE1210
260 DO 270 I=1,N	SLVE1220
N1=J1+I	SLVE1230
270 S(N1)=-SDOT(I-1,A(I,1),MA,S(J1+1),1,-B(I))	SLVE1240
DO 280 K=1,N	SLVE1250
I=NP1-K	SLVE1260
N1=J1+I	SLVE1270
280 X(I)=-SDOT(N-I,A(I,I+1),MA,X(I+1),1,-S(N1))/A(I,I)	SLVE1280
C	SLVE1290
C ITERATIVE IMPROVEMENT	SLVE1300
C	SLVE1310
290 IF(ITMAX)370,370,300	SLVE1320
300 TOP=0.0	SLVE1330
DO 310 I=1,N	SLVE1340
310 TOP=AMAX1(TOP,ABS(X(I)))	SLVE1350
EPSX=EPS*TOP	SLVE1360
DO 360 IT=1,ITMAX	SLVE1370
C FIND RESIDUALS	SLVE1380
DO 320 I=1,N	SLVE1390
320 S(I)=-DOT(N,AA(I,1), N,X(1),1,-B(I))	SLVE1400
C FIND INCREMENT	SLVE1410
DO 330 I=1,N	SLVE1420
N1=J1+I	SLVE1430
330 S(N1)=-SDOT(I-1,A(I,1),MA,S(J1+1),1,-S(I))	SLVE1440
DO 340 K=1,N	SLVE1450
I=NP1-K	SLVE1460
N1=J1+I	SLVE1470
340 S(I)=-SDOT(N-I,A(I,I+1),MA,S(I+1),1,-S(N1))/A(I,I)	SLVE1480
C INCREMENT AND TEST CONVERGENCE	SLVE1490
IUP=0.0	SLVE1500
DO 350 I=1,N	SLVE1510
TEMP=X(I)	SLVE1520
X(I)=DAD(X(I),S(I))	SLVE1530
DELX=ABS(X(I)-TEMP)	SLVE1540
TOP=AMAX1(TOP,DELX)	SLVE1550
350 CONTINUE	SLVE1560
IF(TOP-EPSX)380,380,360	SLVE1570
360 CONTINUE	SLVE1580
370 IT=0	SLVE1590
380 RETURN	SLVE1600
C	SLVE1610
C RESTORE A AND B	SLVE1620
C	SLVE1630
390 CONTINUE	SLVE1640
DO 420 K=1,N	SLVE1650
I=NP1-K	SLVE1660
N3=J3+I	SLVE1670
IP=S(N3)	SLVE1680
IF(I-IP)400,420,400	SLVE1690
400 TEMP=B(I)	SLVE1700
B(I)=B(IP)	SLVE1710
B(IP)=TEMP	SLVE1720
DO 410 J=1,N	SLVE1730
TEMP=AA(I,J)	SLVE1740
AA(I,J)=AA(IP,J)	SLVE1750
410 AA(IP,J)=TEMP	SLVE1760
420 CONTINUE	SLVE1770
DO 430 I=1,N	SLVE1780
N2=J2+I	SLVE1790

# Printout D-2 (contd)

B(I)=B(I)/S(N2)	SLVE1800
DU 430 J=1,N	SLVE1810
A(I,J)=AA(I,J)/S(N2)	SLVE1820
430 CONTINUE	SLVE1830
RETURN	SLVE1840
END	SLVE1850
\$IBFTC SLVIN. DECK	SLVI0010
SUBROUTINE SLVINV (NDIM,N,A,B,IT,S)	SLVI0020
C	SLVI0030
C THE DIMENSION OF S MUST BE AT LEAST N**2+5*N	SLVI0040
C	SLVI0050
DIMENSION B(NDIM,NDIM),S(N)	SLVI0060
ITMAX = 10	SLVI0070
IN = 1	SLVI0080
JJ = 4*N	SLVI0090
DO 20 J = 1,N	SLVI0100
DO 10 I = 1,N	SLVI0110
K = JJ + I	SLVI0120
10 S(K) = 0.0	SLVI0130
K = JJ + J	SLVI0140
S(K) = 1.0	SLVI0150
CALL SOLVE(NDIM,A,N,S(4*N+1),B(1,J),IN,7.0E-9,ITMAX,IT,S(5*N+1),S)	SLVI0160
IF (IT.EQ.(-1)) GO TO 30	SLVI0170
IF (IT.EQ.0) CALL SOLVE (NDIM,A,N,S(4*N+1),B(1,J),5,7.0E-9,ITMAX,	SLVI0180
1 IT,S(5*N+1),S)	SLVI0190
20 IN = 2	SLVI0200
CONTINUE	SLVI0210
C	SLVI0220
C RESTORE THE MATRIX A	SLVI0230
C	SLVI0240
30 CALL SOLVE(NDIM,A,N,S(4*N+1),B(1,J), 3,7.0E-9,ITMAX,IT,S(5*N+1),S)	SLVI0250
RETURN	SLVI0260
END	SLVI0270



## Nomenclature

### Coordinates and Related Quantities

$x, y, z$	Cartesian coordinates (basic coordinate system with the origin located at the source of the incident electromagnetic field)
$P$	field point (the point at which the scattered or resultant field is evaluated)
$\mathbf{R}$	the vector from the origin to $P$
$R, \Theta, \Phi$	spherical coordinates of $P$
$S$	the scattering surface
$\rho$	the vector from the origin to a point on $S$
$\rho, \theta, \phi$	spherical coordinates of a point on $S$
$\rho(\theta, \phi)$	a function describing $S$
$\Delta S_{mn}$	an incremental area on $S$
$r$	the distance from $P$ to a point on $S$
$\mathbf{n}$	a unit vector normal to $S$
$\mathbf{i}$	a unit vector in the direction indicated by a coordinate subscript
$\beta$	feed offset angle
$V$	a region of space

### Electromagnetic Quantities

$\mathbf{E}$	electric field
$\mathbf{H}$	magnetic field
$\mathbf{E}_i, \mathbf{H}_i$	incident field
$\mathbf{E}_s, \mathbf{H}_s$	scattered field
$\mathbf{E}_T$	total resultant field
$H_\rho, H_\theta, H_\phi$	components of $\mathbf{H}_i$ with radial variation factored out
$E_\theta, E_\phi$	components of $\mathbf{E}_i$ with radial variation factored out
$E_\theta, E_\phi$	components of $\mathbf{E}_s$ with radial variation factored out
$\mathbf{K}$	surface currents
$\Delta \mathbf{K}$	difference between true currents and physical optics currents
$k = \omega(\epsilon\mu)^{1/2} = 2\pi/\lambda$	the propagation constant

$\epsilon$	electric permittivity
$\lambda$	wavelength
$\mu$	magnetic permeability
$\omega$	angular frequency

### Functions and Constants Used in Derivations

$\mathbf{I}$	a vector function related to $\mathbf{E}_s$
$\mathbf{I}_\theta, \mathbf{I}_\phi$	components of $\mathbf{I}$
$\Delta \mathbf{I}_{mn}$	incremental contribution to $\mathbf{I}$ from $\Delta S_{mn}$
$a_m(\theta), b_m(\theta)$	Fourier components of surface data
$a'_{omn}, b'_{omn}$	coefficients related to the spherical-wave coefficients
$a_n = a_{omn}^e$ $b_n = b_{omn}^e$	spherical-wave coefficients
$\mathbf{n}_n = \mathbf{n}_{omn}^e$ $\mathbf{m}_n = \mathbf{m}_{omn}^e$	spherical-wave functions
$\gamma$	the path-length function
$\mathbf{F}$	a vector function related to $H_i$ and $\rho$
$a_{mn}, b_{mn}, c_{mn}$	interpolation coefficients for the function $\mathbf{F}$
$\alpha_{mn}, \beta_{mn}, \xi_{mn}$	interpolation coefficients for the function $\gamma$
$A_{m_o}^e(\theta) = A(\theta)$ $B_{m_o}^e(\theta) = B(\theta)$	$m$ th even or odd Fourier component of incident electric field

### Miscellaneous Symbols

$j$	$(-1)^{1/2}$
$\nabla$	Laplacian operator
$h_n(k\rho)$	spherical Bessel function
$P_n^m(\cos \theta)$	associated Legendre function
$f$	focal length of a paraboloid
$D$	physical diameter of source (also used as diameter of a scattering surface)
$F_n^m(\theta), G_n^m(\theta)$	functions related to the associated Legendre function

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